18-791 Lecture #7
FREQUENCY RESPONSE OF LSI SYSTEMS

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Introduction

- **Last week we discussed the *Z*-transform at length, including the unit sample response, ROC, inverse *Z*-transforms and comparison to the DTFT and difference equations**
- **Today we will discuss the frequency response of LSI systems and how it relates to the system function in *Z*-transform form**
- **Specifically we will**
  - —Relate magnitude and phase of DTFT to locations of poles and zeros in *z*-plane
  - —Discuss several important special cases:
    - All-pass systems
    - Minimum/maximum-phase systems
    - Linear phase systems

Review - Difference equations and *Z*-transforms characterizing LSI systems

$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$

- **Many LSI systems are characterized by difference equations of the form** $\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$
- **They produce system functions of the form**

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

- **Comment: This notation is a little different from last week s (but consistent with the text in Chap, 5)**

Difference equations and *Z*-transforms characterizing LSI systems (cont.)

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

- **Comments:**
  - —LSI systems characterized by difference equations produce *z*-transforms that are ratios of polynomials in z or z^{-1}
  - —The **zeros** are the values of z that cause the numerator to equal zero, and the **poles** are the values of z that cause the denominator polynomial to equal zero

Discrete-time Fourier transforms and the *Z*-transform

- **Recall that the DTFT is obtained by evaluating the *z*-transform along the contour** $z = e^{j\omega}$

$$H(e^{j\omega}) = H(z)\big|_{z=e^{j\omega}}$$

- **The DTFT is generally complex and typically characterized by its magnitude and phase:**

$$H(e^{j\omega}) = \frac{\sum_{k=0}^{M} b_k e^{-j\omega k}}{\sum_{k=0}^{N} a_k e^{j\omega k}} = \left|H(e^{j\omega})\right| e^{j\angle H(e^{j\omega})}$$

Obtaining the magnitude and phase of the DTFT by factoring the *z*-transform

- **Factoring the *z*-transform:**

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=1}^{M} (1 - c_k z^{-1})}{\prod_{k=1}^{N} (1 - d_k z^{-1})}$$

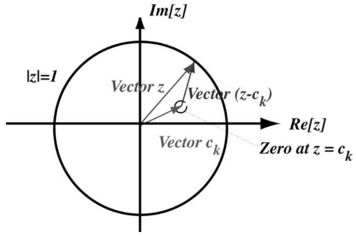
- **Comment: The constants** c_k **and** d_k **are the zeros and poles of the system respectively**

So what do those terms mean, anyway?

- **Convert into a polynomial in z by multiplying numerator and denominator by largest power of z:**

$$H(z) = \left(\frac{b_0}{a_0}\right)\frac{\prod_{k=1}^{M}(1-c_k z^{-1})}{\prod_{k=1}^{N}(1-d_k z^{-1})} = z^{N-M}\left(\frac{b_0}{a_0}\right)\frac{\prod_{k=1}^{M}(z-c_k)}{\prod_{k=1}^{N}(z-d_k)}$$

- **Now consider one of the numerator terms,** $(z-c_k)$



Note that the vector $(z\text{-}c_k)$ is the length of line from the zero to the current value of z or the distance from the zero to the unit circle.

Finding the magnitude of the DTFT

$$H(e^{j\omega}) = H(z)\big|_{z=e^{j\omega}} = z^{N-M}\left(\frac{b_0}{a_0}\right)\frac{\prod_{k=1}^{M}(z-c_k)}{\prod_{k=1}^{N}(z-d_k)}\Bigg|_{z=e^{j\omega}}$$

- **Magnitude:**

$$\left|H(e^{j\omega})\right| = \left|H(z)\right|\big|_{z=e^{j\omega}} = \left|z^{N-M}\right|\left|\frac{b_0}{a_0}\right|\left|\frac{\prod_{k=1}^{M}(z-c_k)}{\prod_{k=1}^{N}(z-d_k)}\right|_{z=e^{j\omega}} = \left|\frac{b_0}{a_0}\right|\frac{\prod_{k=1}^{M}\left|(z-c_k)\right|}{\prod_{k=1}^{M}\left|(z-d_k)\right|}\Bigg|_{z=e^{j\omega}}$$

- **Comment:** The magnitude is the product of magnitudes from zeros divided by product of magnitudes from poles

Finding the phase of the DTFT

$$H(e^{j\omega}) = H(z)\big|_{z=e^{j\omega}} = z^{N-M}\left(\frac{b_0}{a_0}\right)\frac{\prod_{k=1}^{M}(z-c_k)}{\prod_{k=1}^{N}(z-d_k)}\Bigg|_{z=e^{j\omega}}$$

- **Phase:**

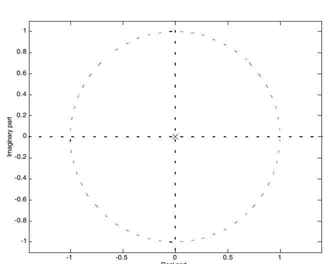
$$\angle H(e^{j\omega})\big|_{z=e^{j\omega}} = \omega(N-M) + \angle\left(\frac{b_0}{a_0}\right) + \sum_{k=1}^{M}\angle(z-c_k) - \sum_{k=1}^{N}\angle(z-d_k)\Bigg|_{z=e^{j\omega}}$$

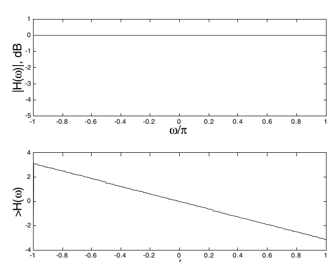
- **Comment:** The magnitude is the sum of the angles from the zeros minus the sums of the angles from the poles

Example 1: Unit time delay

$H(z) = z^{-1}$

- **Pole-zero pattern:** **Frequency response:**

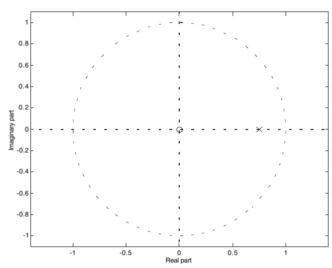


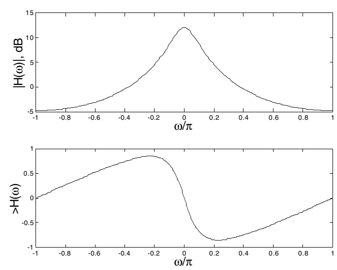


Example 2: Decaying exponential sample response

$$H(z) = \frac{1}{1 - \alpha z^{-1}}$$

- **Pole-zero pattern:** **Frequency response:**

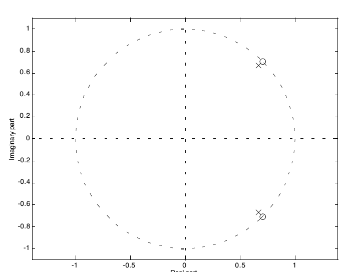


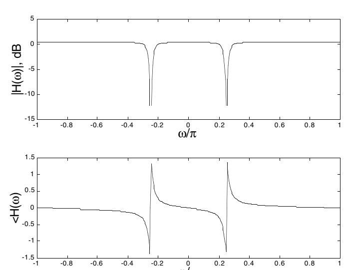


Example 3: Notch filter

$$H(z) = \frac{(z - e^{j\pi/4})(z - e^{-j\pi/4})}{(z - .95e^{j\pi/4})(z - .95e^{-j\pi/4})}$$

- **Pole-zero pattern:** **Frequency response:**





Summary (first half)

- **The DTFT is obtained by evaluating the *z*-transform along the unit circle**
- **As we walk along the unit circle,**
  - —The magnitude of the DTFT is proportional of the product of the distances from the zeros divided by the product of the distances from the poles
  - —The phase of the DTFT is (within additive constants) the sum of the angles from the zeros minus the sum of the angles from the poles
- **After the break:**
  - —Allpass systems
  - —Minimum-phase and maximum-phase systems
  - —Linear-phase systems

Special types of LSI systems

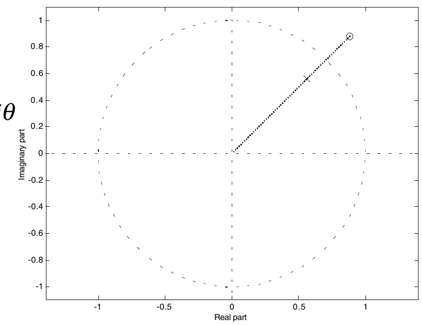
- **We can get additional insight about the frequency-response behavior of LSI systems by considering three special cases:**
  - —Allpass systems
  - —Systems with minimum or maximum phase
  - —Linear-phase systems

All-pass systems

- **Consider an LSI system with system function** $H(z) = \dfrac{z^{-1} - \alpha^*}{1 - \alpha z^{-1}}$ **with α complex ..**
- **Let** $\alpha = re^{j\theta}$

  —Then there is a pole at $z = re^{j\theta}$

  —And a zero at $z = \dfrac{1}{r} e^{j\theta}$



- **Comment: We refer to this configuration as mirror image poles and zeros**

Frequency response of all-pass systems

$$H(z) = \frac{z^{-1} - \alpha^*}{1 - \alpha z^{-1}}; \quad z = e^{j\omega}, \alpha = re^{j\theta}$$

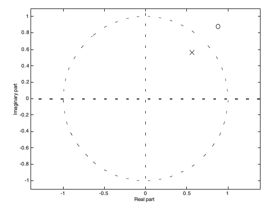
- **Obtaining magnitude of frequency response directly:**

$$\left|H(e^{j\omega})\right|^2 = H(e^{j\omega})H^*(e^{j\omega}) = \frac{(e^{-j\omega} - re^{-j\theta})}{(1 - re^{j\theta}e^{-j\omega})} \frac{(e^{j\omega} - re^{j\theta})}{(1 - re^{-j\theta}e^{j\omega})}$$

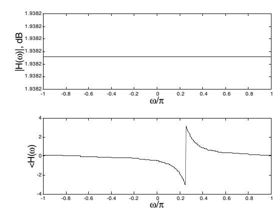
$$= \frac{(1 - re^{j(\theta - \omega)} - re^{-j(\theta - \omega)} + r^2)}{(1 - re^{-j(\theta - \omega)} - re^{j(\theta - \omega)} + r^2)} = 1$$

Frequency response of all-pass systems

- All-pass systems have mirror-image sets of poles and zeros



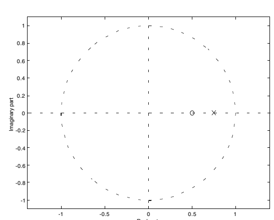
- All-pass systems have a frequency response with constant magnitude

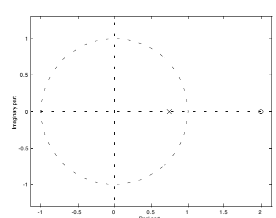


System functions with the same magnitude can have more than one phase function

- Consider two systems:

System 1: pole at .75, zero at .5   System 2: pole at .75, zero at 2

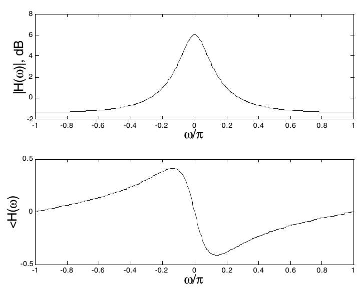




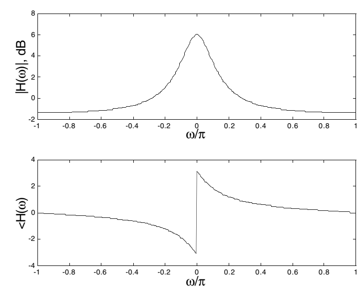
- Comment: System 2 can be obtained by cascading System 1 with an all-pass system with a pole at .5 and a zero at 2. Hence the two systems have the same magnitude.

But what about the two phase responses?

Response of System 1:



Response of System 2:



- **Comment: Systems have same magnitude, but System 2 has much greater phase shift**

General comments on phase responses

- **System 1 has much less phase shift than System 2; this is generally considered to be good**
- **System 1 has its zero inside unit circle; System 2 has zero its zero outside the unit circle**
- **A system is considered to be of minimum phase if all of its zeros lie inside the unit circle**
- **A system is considered to be of maximum phase if all of its zeros lie outside the unit circle**
- **Systems with more than one zero might have neither minimum nor maximum phase**

A digression: Symmetry properties of DTFTs

- **Recall from DTFT properties:**

- **If** $x[n] \Leftrightarrow X(e^{j\omega})$

then $x[-n] \Leftrightarrow X(e^{-j\omega})$

$x^*[n] \Leftrightarrow X^*(e^{-j\omega})$

and ... $x[n] = x[-n] \Rightarrow X(e^{j\omega}) = X(e^{-j\omega})$

even — even

$x[n] = -x[-n] \Rightarrow X(e^{j\omega}) = -X(e^{-j\omega})$

odd — odd

$x[n] = x^*[n] \Rightarrow X(e^{j\omega}) = X^*(e^{-j\omega})$

real — Hermitian symmetric

Consequences of Hermitian symmetry

- **If** $X(e^{j\omega}) = X^*(e^{-j\omega})$

Then

$\text{Re}[X(e^{j\omega})]$ is even

$\text{Im}[X(e^{j\omega})]$ is odd

$|X(e^{j\omega})|$ is even

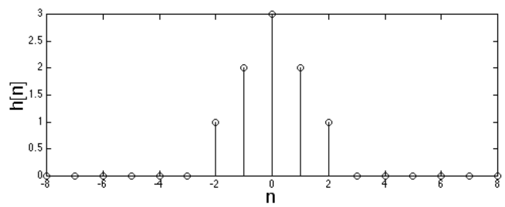
$\angle X(e^{j\omega})$ is odd

And If x[n] is real and even, $X(e^{j\omega})$ will be real and even

and if x[n] is real and odd, $X(e^{j\omega})$ will be imaginary and odd

Zero phase systems

- **Consider an LSI system with an even unit sample response:**



- **DTFT is** $H(z) = e^{2j\omega} + 2e^{j\omega} + 3 + 2e^{-j\omega} + e^{-2j\omega}$

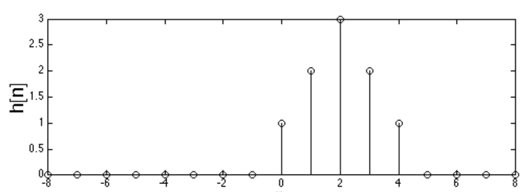
$$= 2\cos(2\omega) + 4\cos(\omega) + 3$$

- **Comments:**
  —Frequency response is real, so system has zero phase shift

  —This is to be expected since unit sample response is real and even

Linear phase systems

- **Now delay the system s sample response to make it causal:**



- **DTFT is now** $H(z) = e^{2j\omega} + 2e^{j\omega} + 3 + 3e^{-j\omega} + e^{-2j\omega}$

$$= e^{-2j\omega}(e^{2j\omega} + 2e^{j\omega} + 3 + 2e^{-j\omega} + e^{-2j\omega})$$

$$= e^{-2j\omega}(2\cos(2\omega) + 4\cos(\omega) + 3)$$

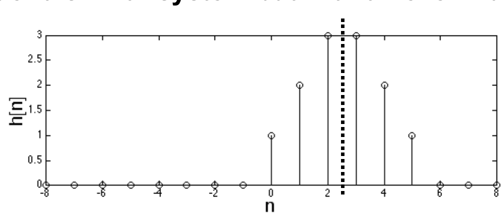
- **Comment:**
  —Frequency response now exhibits **linear phase shift**

An additional comment or two

- The system on the previous page exhibits linear phase shift
- This is also reasonable, since the corresponding sample response can be thought of as a zero-phase sample response that undergoes a time shift by two samples (producing a linear phase shift in the frequency domain)
- Another way to think about this is as a sample response that is even symmetric about the sample *n=2*
- Linear phase is generally considered to be more desirable than non-linear phase shift
- If a linear-phase system is causal, it must be finite in duration. (The current example has only 5 nonzero samples.)

Another example of a linear phase systems

- Now let s consider a similar system but with an even number of sample points:



- DTFT is

$$H(z) = e^{j\omega} + 2e^{-j\omega} + 3e^{-2j\omega} + 3e^{-3j\omega} + 3e^{-4j\omega} + e^{-5j\omega}$$

$$= e^{-2.5j\omega}(e^{2.5j\omega} + 2e^{1.5j\omega} + 3e^{.5j\omega} + 3e^{-.5j\omega} + 2e^{-1.5j\omega} + e^{-2.5j\omega})$$

$$= e^{-2.5j\omega}(2\cos(2.5\omega) + 4\cos(1.5\omega) + 3\cos(.5\omega))$$

Comments on the last system

- The system on the previous page also exhibits linear phase shift
- In this case the corresponding sample response can be thought of as a zero-phase sample response that undergoes a time shift by 2.5 samples
- In this case the unit sample response is symmetric about the point $n=2.5$
- This type of system exhibits generalized linear phase, because the unit sample response is symmetric about a location that is between two integers

Four types of linear-phase systems

- Oppenheim and Schafer refer to four types systems with generalized linear phase. All have sample points that are symmetric about its midpoint.
  - —Type I: Odd number of samples, even symmetry
  - —Type II: Even number of samples, even symmetry
  - —Type III: Odd number of samples, odd symmetry
  - —Type IV: Even number of samples, odd symmetry

Summary of second half of lecture

- All-pass systems have poles and zeros in mirror-image pairs
- Minimum phase causal and stable systems have all zeros (as well as all poles) inside the unit circle
- Maximum phase causal and stable systems have all zeros outside the unit circle
- Linear phase systems have unit sample responses that are symmetric about their midpoint (which may lie between two sample points