Carnegie Mellon DSP-I Electrical & Computer ENGINEERING

Digital Signal Processing I (18-791)
Fall Semester, 2005

EXAMPLES OF CONVOLUTION COMPUTATION

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Introduction

These notes briefly review the convolution examples presented in the recitation section of September 3.

Computation of the convolution sum – Example 1

As I mentioned in the recitation, it is important to understand the convolution operation on many levels. We use graphical representations of the functions in the convolution sum (as demonstrated in class using MATLAB) to give us overall insight into the form of the output and the limits of non-zero output points. In this section we will provide an example of how the convolution sum is computed analytically.

Computation of actual convolution sums in 18-791 most commonly involves generalized exponential functions (including step functions and the sine/cosine functions). Two mathematical relationships that are frequently used are

$$\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha}, \text{ which converges only for } |\alpha| < 1, \text{ and}$$

$$\sum_{n=0}^{N-1} \alpha^n = \frac{1-\alpha^N}{1-\alpha}, \text{ which is a finite sum and hence always converges}$$

Now let's consider the following pair of functions to be convolved:

$$x[n] = u[n] \text{ and}$$

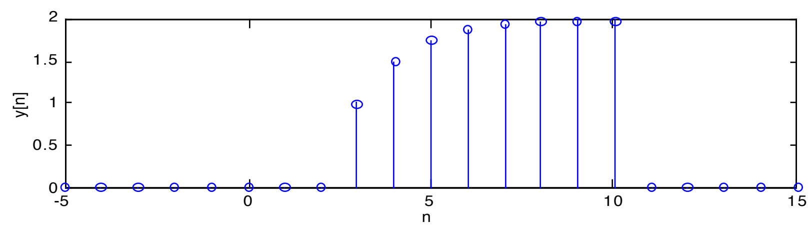
$$h[n] = \left(\frac{1}{2}\right)^{n-3} u[n-3]$$

The graphical convolution of these function is illustrated in the MATLAB routines that are available in

`http://www.ece.cmu.edu/~ee791/lectures/L02`

The calling scripts `go4.m` and `go5.m` flip and shift $x[n]$ and $h[n]$, respectively (although the notation on the screens may be a little different).

Note that either way, the result of the convolution looks like



To obtain this result analytically we must first decide which function to flip and shift. Normally we'd flip and shift the "simpler" time function, which in this case would be $x[n]$, causing the convolution sum to be written as

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

Strictly for the purpose of being more illustrative, though, we will flip and shift the more complicated function $h[n]$, causing the convolution sum to be written as

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Given our definitions above for $x[n]$ and $h[n]$, we can write by substitution

$$h[n-k] = \left(\frac{1}{2}\right)^{n-k-3} u[n-k-3]$$

$$x[k] = u[k]$$

Since $x[k]$ equals zero for $x < 0$ and $h[n-k]$ equals zero for $k > n-3$, we can see that the only region of "overlap" for which both $x[k]$ and $h[n-k]$ are nonzero is $0 \le k \le n-3$. Hence the convolution sum is equal to zero for values of n that are less than 3. For $3 \le n$ we have

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=0}^{n-3} \left(\frac{1}{2}\right)^{n-k-3}$$

$$y[n] = \left(\frac{1}{2}\right)^{n-3} \sum_{k=0}^{n-3} 2^k = \left(\frac{1}{2}\right)^{n-3} \frac{(1-2^{n-2})}{(1-2)}$$

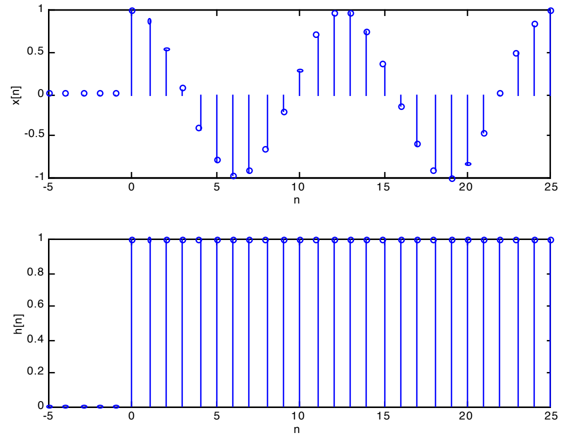
$$y[n] = \left(\frac{1}{2}\right)^{n-3} \left(\left(\frac{1}{2}\right)^{2-n} - 1\right) = 2 - \left(\frac{1}{2}\right)^{n-3}$$

So, $$y[n] = \left(2 - \left(\frac{1}{2}\right)^{n-3}\right) u[n-3]$$

Computation of the convolution sum – Example 2

$x[n]$

$h[n]$



Now consider the convolution of $x[n] = \cos(\omega_0 n)u[n]$ with $h[n] = u[n]$.

First, let's recall the three "tricks" to be invoked that were discussed in the recitation:

- **Trick 1.** Representing sines and cosines in complex exponential form:

$$\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$\sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

- **Trick 2.** Infinite and finite summation of exponentials:

$$\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha}, \text{ for } |\alpha| < 1$$

$$\sum_{n=0}^{N-1} \alpha^n = \frac{1-\alpha^N}{1-\alpha}, \text{ for all } \alpha$$

- **Trick 3.** Balancing equations with complex exponentials:

$$(1 - e^{j\omega n}) = e^{j\omega n/2}(e^{-j\omega n/2} - e^{j\omega n/2}) = -2je^{j\omega n/2}\sin(\omega/2)$$

As noted in class, $y[n] = 0$ for $n < 0$. For $0 \le n \le N-1$ we obtain using the convolution sum

$$y[n] = \sum_{k=0}^{n} x[k]h[n-k] = \sum_{k=0}^{n} \cos(\omega_0 k)$$

Applying Trick 1, we obtain

$$y[n] = \sum_{k=0}^{n} \cos(\omega_0 k) = \frac{1}{2}\sum_{k=0}^{n} e^{j\omega_0 k} + \frac{1}{2}\sum_{k=0}^{n} e^{-j\omega_0 k}$$

Applying Trick 2 (not worrying about convergence because the sum is finite), we obtain

$$y[n] = \frac{1}{2}\sum_{k=0}^{n} e^{j\omega_0 k} + \frac{1}{2}\sum_{k=0}^{n} e^{-j\omega_0 k} = \frac{1}{2}\frac{\left(1 - e^{j\omega_0(n+1)}\right)}{\left(1 - e^{j\omega_0}\right)} + \frac{1}{2}\frac{\left(1 - e^{-j\omega_0(n+1)}\right)}{\left(1 - e^{-j\omega_0}\right)}$$

Applying Trick 3, we obtain

$$y[n] = \frac{1}{2}\frac{e^{j\omega_0\left(\frac{n+1}{2}\right)}\left(e^{-j\omega_0\left(\frac{n+1}{2}\right)} - e^{j\omega_0\left(\frac{n+1}{2}\right)}\right)}{e^{j\omega_0/2}\left(e^{-j\omega_0/2} - e^{j\omega_0/2}\right)} + \frac{1}{2}\frac{e^{-j\omega_0\left(\frac{n+1}{2}\right)}\left(e^{j\omega_0\left(\frac{n+1}{2}\right)} - e^{-j\omega_0\left(\frac{n+1}{2}\right)}\right)}{e^{-j\omega_0/2}\left(e^{j\omega_0/2} - e^{-j\omega_0/2}\right)}$$

$$= \frac{1}{2}\frac{e^{j\omega_0 n/2} - 2j\sin\left(\frac{\omega_0(n+1)}{2}\right)}{-2j\sin\left(\frac{\omega_0}{2}\right)} + \frac{1}{2}\frac{e^{-j\omega_0 n/2}\, 2j\sin\left(\frac{\omega_0(n+1)}{2}\right)}{2j\sin\left(\frac{\omega_0}{2}\right)}$$

Combining terms, we obtain

$$y[n] = \frac{\sin\left(\frac{\omega_0(n+1)}{2}\right)}{\sin\left(\frac{\omega_0}{2}\right)}\cos\left(\frac{\omega_0 n}{2}\right) \quad \text{for } 0 \le n$$

Note that this result *must* be real, as the two functions in the convolution sum are real.