18-550

SOLUTIONS TO TEST # 2 FALL 1999

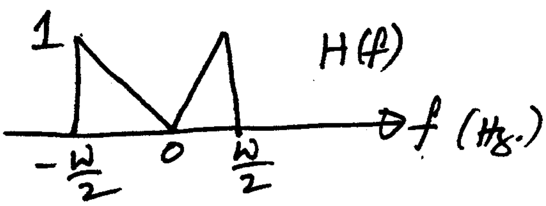
PROBLEM 1

R_s, Symbol rate $= 4000 \dfrac{\text{samples}}{\text{sec}} \times \dfrac{5 \text{ bits}}{\text{sample}} \times \dfrac{1 \text{ symbol}}{4 \text{ bits}}$

$= 5000$ symbols/sec.

For zero ISI, we must satisfy Nyquist condition, i.e.,

$$\sum_{K=-\infty}^{\infty} H(f - KR_s) \quad \text{must be a constant.}$$



By inspection $\sum_{K=-\infty}^{\infty} H(f - K\frac{W}{2})$ is constant.

$\therefore \quad R_s = \dfrac{W}{2} \Rightarrow W = 2R_s = 10{,}000$ ~~symbols/sec~~ Hz.

PROBLEM 2

By inspection, the 4 signals $s_i(t)$, $i = 1,2,3,4$ are orthogonal.

Also, the energy of each of the signals is same, i.e.,

$$E_s = \int_{-\infty}^{\infty} s_i^2(t)\,dt = 1\,(10^{-3}) = 10^{-3}.$$

Since there are 4 symbols, we have 2 bits for each symbol.

$\therefore \quad E_b = \frac{1}{2} E_s = 5 \times 10^{-4}$

$N_0 = 10^{-4} \Rightarrow \dfrac{E_b}{N_0} = 5. \qquad \dfrac{E_s}{N_0} = 10.$

Since the 4 signals are equally likely, of equal energy and are orthogonal, we can use the P_B expressions given for M-ary orthogonal schemes in eqs. (3.123) and (3.127).

$$P_E(4) = \frac{1}{4}\exp\left(-\frac{E_S}{N_0}\right)\cdot\sum_{j=2}^{4}(-1)^j\binom{4}{j}\exp\left(\frac{E_S}{jN_0}\right)$$

$$= \frac{1}{4}\cdot e^{-10}\left\{6\cdot e^{5} - 4e^{10/3} + e^{10/4}\right\}$$

$$= \frac{1}{4}\left\{6\cdot e^{-5} - 4\cdot e^{-20/3} + e^{-30/4}\right\} = 0.009$$

$$\frac{P_B}{P_E} = \frac{M/2}{M-1} = \frac{2}{3} \Rightarrow P_B = \frac{2}{3}P_E = 0.006$$

PROBLEM 3 UNCODED

$\frac{E_b}{N_0} = \frac{S}{N_0 R_b} = \frac{8000}{1000} = 8$; $\frac{E_b}{N_0} = 9\text{ dB}$ OR $\frac{E_b}{N_0} = 8$ (3 dB means $\frac{E_b}{N_0} = 2$).

For coherent BPSK, $p_u = Q\left(\sqrt{\frac{2E_b}{N_0}}\right) = Q(4) =$

$$\cong \frac{1}{x\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right) = \frac{1}{4\sqrt{2\pi}}\exp(-8) = 3.3458\times10^{-5}$$

The probability that the uncoded message block will be received in error is

$$P_M^u = 1-(1-p_u)^{12} = 4.0142\times10^{-4}$$

For (23,12) CODED TEC

$$R_c = R_b\cdot\frac{23}{12} = \frac{23000}{12}\text{ bits/sec}.$$

$$\frac{E_c}{N_0} = \frac{S}{N_0 R_c} = \frac{(8000)(12)}{23000} = \frac{96}{23}$$

$$\therefore\ p_c = Q\left(\sqrt{\frac{2E_c}{N_0}}\right) = Q\left(\sqrt{\frac{192}{23}}\right) = Q(2.889) = 0.0019$$

(From Page 742)

For (23, 12) TEC code,

$$P_M^c = \sum_{j=4}^{23} \binom{23}{j} (p_c)^j (1-p_c)^{(23-j)}$$

$$\cong \binom{23}{4} p_c^4 (1-p_c)^{19} \cong 1.113 \times 10^{-7}$$

$$\therefore\ \frac{P_M^U}{P_M^c} = \frac{4.0142 \times 10^{-4}\ \cancel{3.3458 \times 10^{-5}}}{1.113 \times 10^{-7}} \cong \cancel{300}\ 3590$$

PROBLEM 4

Message	Code word
00	0000
01	0101
10	1010
11	1111

$n = 4$
$k = 2$ (since there are $2^k = 4$ code words)

Part (a) Generator Matrix G has two rows. These rows correspond to elemental message vectors 10 and 01.

$$\therefore\ \underline{G} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{matrix} \uparrow \\ 2 \\ \downarrow \end{matrix}$$

$$\longleftarrow 4 \longrightarrow$$

Part (b) From Part (a), $\underline{G} = [\underline{I}_2 \,\vdots\, \underline{I}_2]$

$\therefore \; \underline{P} = \underline{I}_2 .$

$$\underline{H} = [\underline{I}_2 \,\vdots\, \underline{P}^T] = [\underline{I}_2 \,\vdots\, \underline{I}_2] = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

Easy to verify that $\underline{G}\,\underline{H}^T = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Part (c) From the 4 codewords in the table, the smallest weight of a nonzero codeword is 2.

$\therefore \; d_{min} = 2 .$

This code can be used to detect a single error in 4 bits.

PROBLEM 5

$$\underline{G} = \begin{array}{c} \left[\begin{array}{cccc|cccccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \updownarrow 6 \\ \longleftarrow 4 \longrightarrow \;\; \longleftarrow 6 \longrightarrow \\ \longleftarrow 10 \longrightarrow \end{array}$$

Part (a) By inspection, $n = 10, \; k = 6 .$

Part (b) $\underline{G} = [\underline{P} \,\vdots\, \underline{I}_k] \Longrightarrow$

$$\underline{P} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

$$\underline{H} = \left[I_{n-k} \,\vdots\, P^T \right] = \begin{array}{c} \left[\begin{array}{cccc|cccccc} 1&0&0&0&1&1&1&0&1&0 \\ 0&1&0&0&1&0&0&1&1&1 \\ 0&0&1&0&0&1&0&0&1&1 \\ 0&0&0&1&0&0&1&1&1&1 \end{array}\right] \updownarrow 4 \\ \longleftarrow 4 \longrightarrow \longleftarrow 6 \longrightarrow \\ \longleftarrow 10 \longrightarrow \end{array}$$

To find d_{min}, we must find how many columns of $\underline{H}$ add up to ~~zero~~ give a zero vector.

- Since no column of $\underline{H}$ is a zero vector, $\Rightarrow d_{min} > 1$.
- Since no two columns of $\underline{H}$ are identical $\Rightarrow d_{min} > 2$.
- Adding columns numbered 1, 2, 5 gives an all-zero vector $\Longrightarrow d_{min} \leq 3$.

$\therefore$ $d_{min} = 3$; can do single error correction.

Part (c)

$$\underline{r} = [0\,1\,0\,0\,0\,0\,1\,0\,1\,1]$$

$$\therefore \text{Syndrome } \underline{s} = \underline{r}\,\underline{H}^T = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

This $\underline{s}$ is same as the 8-th column of $\underline{H}$

$$\Longrightarrow \quad \underline{e} = [0\,0\,0\,0\,0\,0\,0\,1\,0\,0]$$

error in 8-th bit

$$\therefore \quad \hat{\underline{u}} = \underline{r} + \underline{e} = \underbrace{[0\,1\,0\,0\,0\,0\,1\,1\,1\,1]}_{\text{Valid code word, i.e., the sum of bottom 4 rows of } \underline{G}}$$

PROBLEM 6

$$g(x) = 1+x^3+x^4$$

Since $g(x)$ is a primitive polynomial,

$$n = 2^4-1 = 15.$$

$$\therefore \quad k = 15-4 = 11.$$

$$\underline{m} = [1\,0\,0\,0\,1\,0\,0\,1\,0\,0\,1]$$

$$\Rightarrow \quad m(x) = 1+x^4+x^7+x^{10}$$

Find remainder $r(x)$ by dividing $x^4(x^{10}+x^7+x^4+1)$ by $g(x) = x^4+x^3+1$

$$
\begin{array}{r|l}
 & x^{10}+x^9+x^8+x^6+x^2+x \\
\hline
x^4+x^3+1 & x^{14}+x^{11}+x^8+x^4 \\
 & x^{14}+x^{13}+x^{10} \\
\cline{2-2}
 & x^{13}+x^{11}+x^{10}+x^8+x^4 \\
 & x^{13}+x^{12}+x^9 \\
\cline{2-2}
 & x^{12}+x^{11}+x^{10}+x^9+x^8+x^4 \\
 & x^{12}+x^{11}+x^8 \\
\cline{2-2}
 & x^{10}+x^9+x^4 \\
 & x^{10}+x^9+x^6 \\
\cline{2-2}
 & x^6+x^4 \\
 & x^6+x^5+x^2 \\
\cline{2-2}
 & x^5+x^4+x^2 \\
 & x^5+x^4+x \\
\cline{2-2}
 & x^2+x
\end{array}
$$

$$\therefore \quad x^4(x^{10}+x^7+x^4+1) \bmod (x^4+x^3+1) = x^2+x.$$

$$\therefore \quad u(x) = r(x) + x^4 m(x)$$
$$= x + x^2 + x^4 + x^8 + x^{11} + x^{14}$$

$$\underline{u} = \underbrace{[\underbrace{0\,1\,1\,0}_{4}\,\underbrace{1\,0\,0\,0\,1\,0\,0\,1\,0\,0\,1}_{11}]}_{15}$$