18-550 Fall 1999

SOLUTIONS TO TEST #1

Problem 1

$T_s = 100\ \mu\text{sec}. \qquad f_s = \frac{1}{T_s} = 10\ \text{KHz}.$

64 levels = 6 bits $\Rightarrow$ 6 bits / 100 μsec

or 60 Kbps

Each 16-ary ~~signal~~ symbol can denote 4 bits

Part (a) $\therefore$ symbol rate $= \dfrac{60\ \text{Kbps}}{4\ \text{bits/symbol}}$

$= 15{,}000$ symbols/sec.

Part (b) If signal varies between -1 V and $+1$ V, then 64-level quantization results in

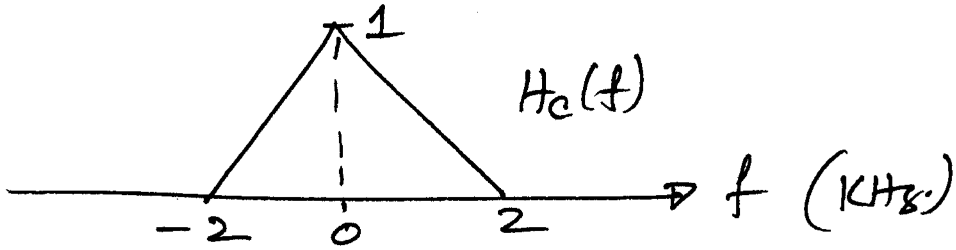
quantization step $= \dfrac{2}{64} = \dfrac{1}{16}$ Volts.

Then the quantization error can be assumed to be uniformly distributed in the interval $\left(-\frac{1}{32}, \frac{1}{32}\right)$ V.

$\therefore$ Variance $= \dfrac{(1/16)^2}{12} = \dfrac{1}{(12)(256)} = \dfrac{1}{3072}$ volt^2

Problem 2

$N_o/2 = 10^{-6}$ watts/Hz.

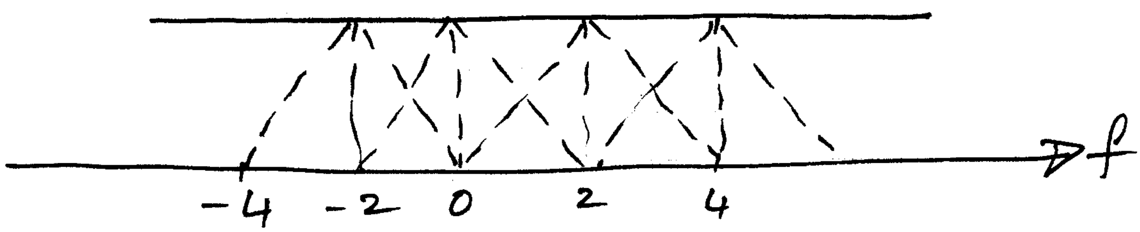


Part (a) $H(f) = H_c(f)\,H_r(f) = H_c(f)$ since $H_r(f) = 1$.

For no ISI, we must satisfy Nyquist condition, i.e.,

$\sum_k H_c\left(f - \frac{k}{T}\right)$ must be a constant.

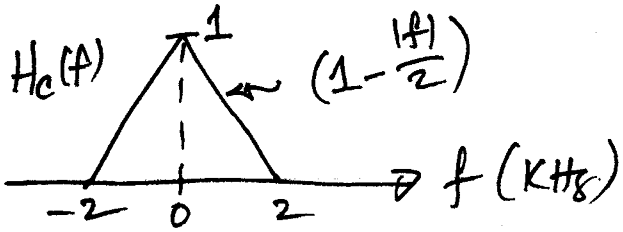
From the following figure, we see that $\sum_k H_c\left(f - \frac{k}{T}\right)$ is a constant if $\frac{1}{T} = 2$ KHz.

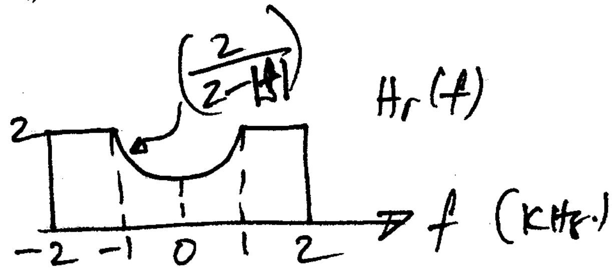


If we try $\frac{1}{T}$ larger than 2 KHz., the sum would not yield a constant.

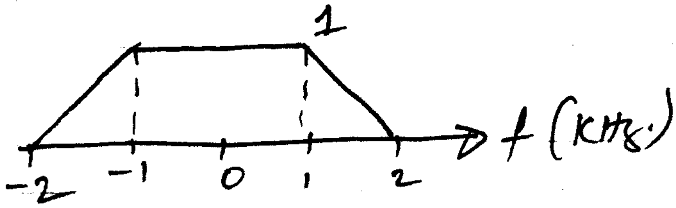
$\therefore$ Maximal Symbol rate with no ISI = 2 KHz.

Part (b) $H(f) = H_c(f)\,H_r(f)$

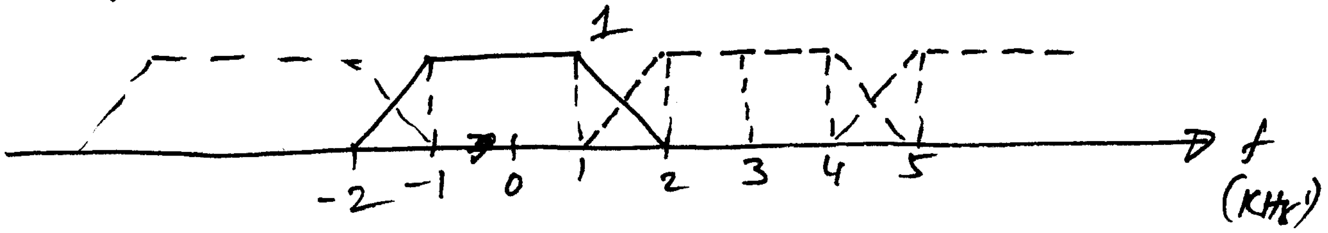




$\therefore$ $H(f) = H_c(f) \cdot H_r(f)$



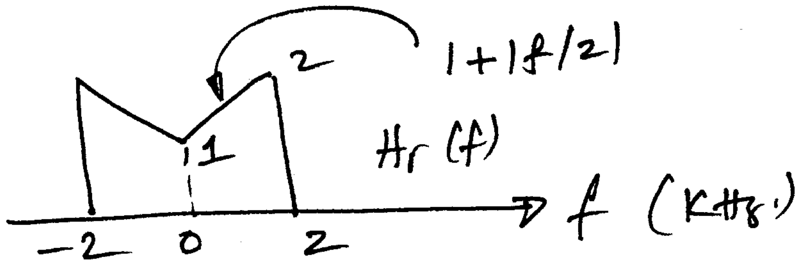
For Nyquist condition, $\sum_K H(f - \frac{K}{T})$ must be constant.



From the above, we see that $\sum_K H(f - \frac{K}{T})$ is a constant if $\frac{1}{T} = 3$ KHz.

$\therefore$ Maximum Symbol rate without ISI = 3 KHz.

Part (c)



Noise goes through the receive filter.

output noise Power Spectral Density $P_n(f) = \frac{N_0}{2}|H_r(f)|^2$

$$P_n(f) = \begin{cases} \frac{N_0}{2}\left(1+\frac{|f|}{2}\right)^2 & \text{for } |f| \le 2 \text{ KHz.} \\ 0 & \text{otherwise.} \end{cases}$$

$$\therefore \text{Variance} = \int_{-\infty}^{\infty} P_n(f)\,df$$

$$= 10^{-6}\int_{-2000}^{2000}\left(1+\frac{|f|}{2000}\right)^2 df = 2\times10^{-6}\int_0^{2000}\left(1+\frac{f}{1000}+\frac{f^2}{4\times10^6}\right)df$$

$$= 2\times10^{-6}\left[f+\frac{f^2}{2000}+\frac{f^3}{12\times10^6}\right]\Bigg|_0^{2000}$$

$$= 2\times10^{-6}\left[2000+2000+\frac{8\times10^9}{12\times10^6}\right]$$

$$= (2)(4667)\times10^{-6} \text{ watts} = 9.334 \text{ mW}.$$

Part (d)

$$T_s = \frac{1}{4000}\ \text{sec} = 250\ \mu\text{sec}.$$

Desired transfer function $H_d(f) = (1+D)\Big|_{D = e^{j2\pi f T_s}}$

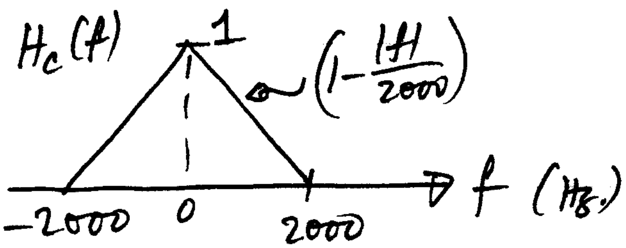
$$= 1 + e^{j2\pi f T_s} = e^{j\pi f T_s}\left(e^{-j\pi f T_s} + e^{j\pi f T_s}\right)$$

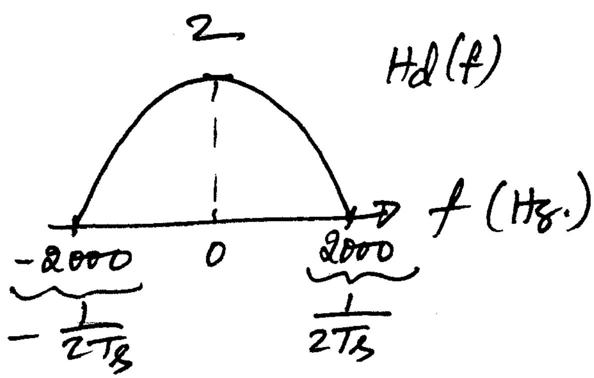
$$= 2e^{j\pi f T_s} \cos(\pi f T_s)$$

$$H_c(f) = \begin{cases} 1 - \frac{|f|}{2000} & \text{for } |f| \le 2000 \\ 0 & \text{for } |f| > 2000 \end{cases}$$

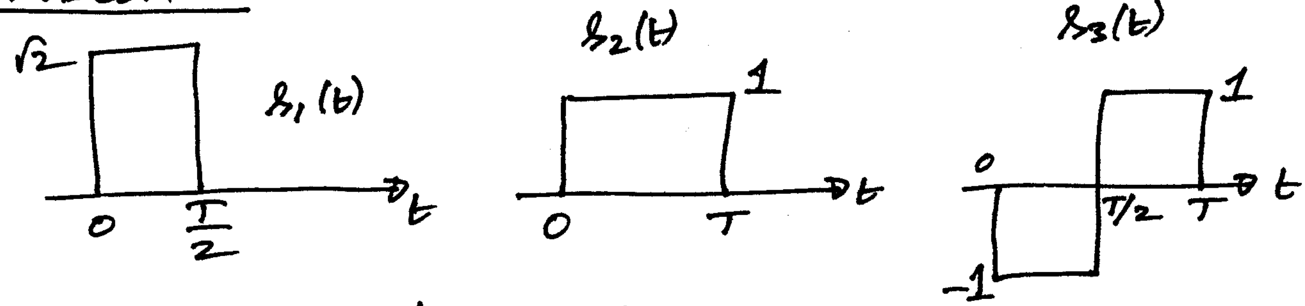
$$\therefore \quad H_r(f) = \frac{H_d(f)}{H_c(f)};$$

$$= \begin{cases} e^{j\pi f (0.25)10^{-3}} \dfrac{2\cos(0.25\pi f \times 10^{-3})}{\left(1 - \frac{|f|}{2000}\right)} & \text{for } |f| \le 2000 \\ 0 & \text{otherwise.} \end{cases}$$





PROBLEM 3

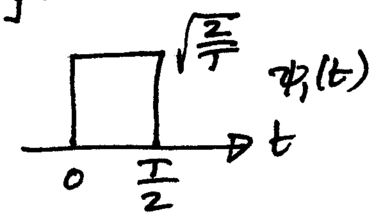


All three signals have same energy, i.e.,

$$\int_0^T s_i^2(t)\,dt = T \qquad i = 1,2,3.$$

Part (a) Use Gram-Schmit Procedure to get orthonormal basis set $\{\psi_1(t), \psi_2(t), \dots, \psi_N(t)\}$.

$$\psi_1(t) = s_1(t)/\|s_1(t)\| = \frac{1}{\sqrt{T}} s_1(t)$$

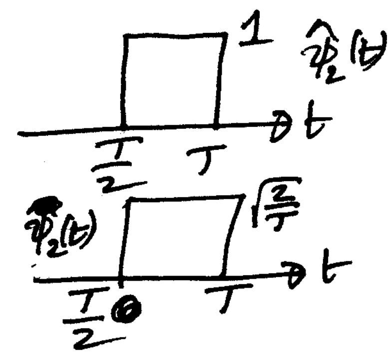


$$\therefore \quad s_1(t) = \sqrt{T}\,\psi_1(t).$$

$$\hat{\psi}_2(t) = s_2(t) - \langle s_2, \psi_1 \rangle \psi_1(t)$$

$$\langle s_2, \psi_1 \rangle = \int_0^{T/2} \left(\sqrt{\tfrac{2}{T}}\right)(1)\,dt = \sqrt{\tfrac{T}{2}}$$

$$\therefore \quad \hat{\psi}_2(t) = s_2(t) - \sqrt{\tfrac{T}{2}}\,\psi_1(t) =$$

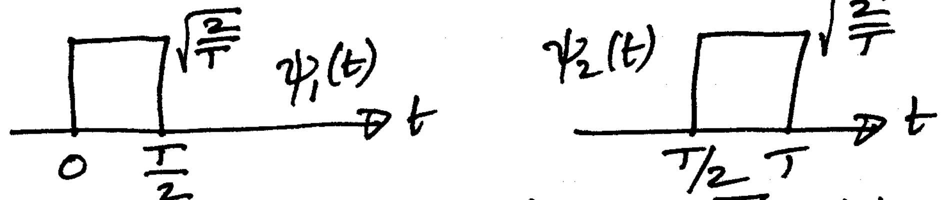


$$\psi_2(t) = \hat{\psi}_2(t)/\|\hat{\psi}_2(t)\| = \sqrt{\tfrac{2}{T}}\,\hat{\psi}_2(t)$$

$$\hat{\psi}_3(t) = s_3(t) - \langle s_3, \psi_1 \rangle \psi_1(t) - \langle s_3, \psi_2 \rangle \psi_2(t)$$

$$= s_3(t) + \sqrt{\tfrac{T}{2}}\,\psi_1(t) - \sqrt{\tfrac{T}{2}}\,\psi_2(t) = 0$$

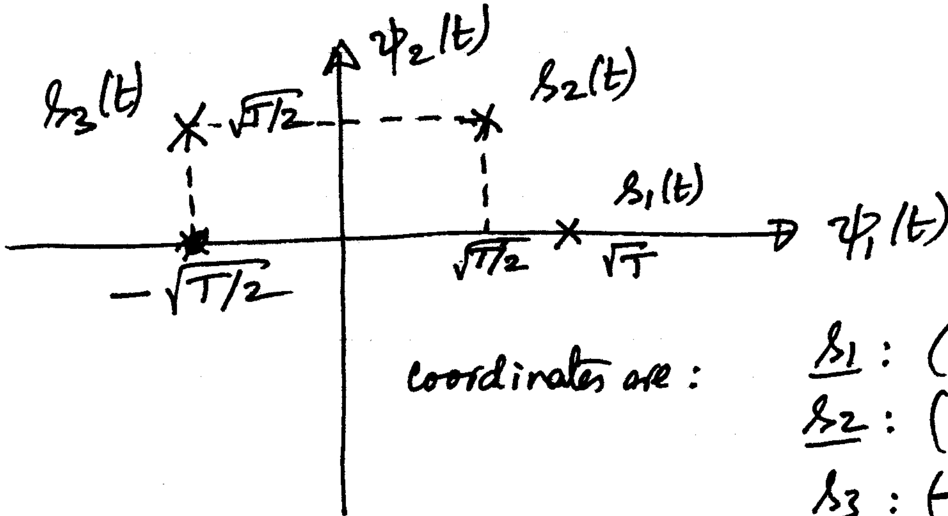
$\therefore$ Need only two orthonormal functions, i.e., $N = 2$.



$$\therefore \quad s_1(t) = \sqrt{T}\,\psi_1(t); \quad s_2(t) = \sqrt{\tfrac{T}{2}}\,\psi_1(t) + \sqrt{\tfrac{T}{2}}\,\psi_2(t);$$

$$s_3(t) = -\sqrt{\tfrac{T}{2}}\,\psi_1(t) + \sqrt{\tfrac{T}{2}}\,\psi_2(t).$$

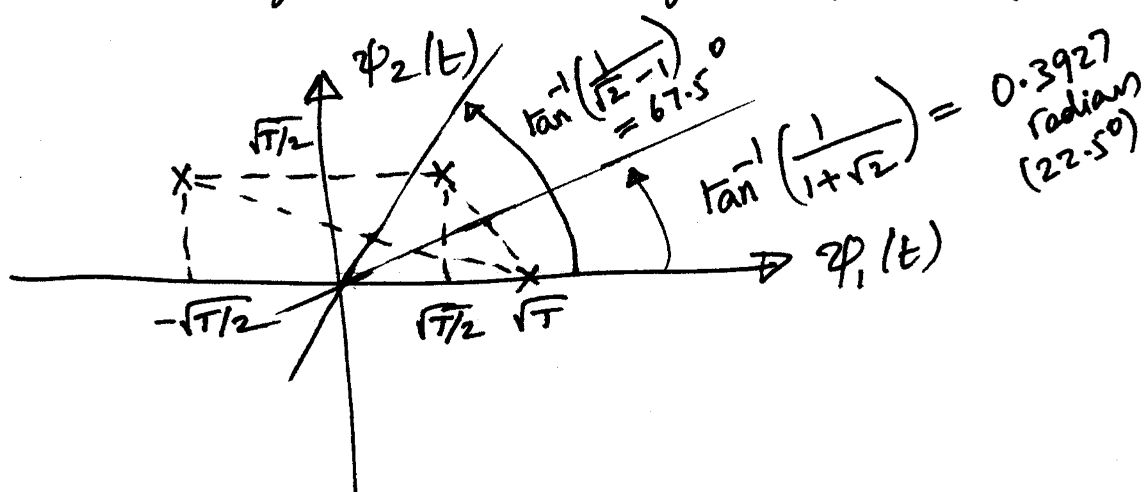
Part (b)

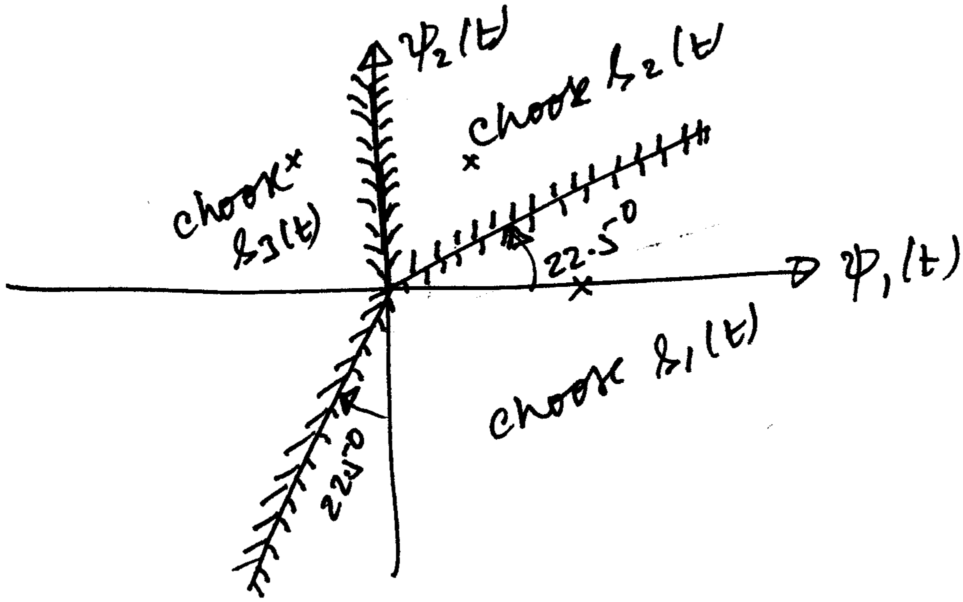


coordinates are : $\underline{s_1}$: $(\sqrt{T}, 0)$
$\underline{s_2}$: $\left(\sqrt{\frac{T}{2}}, \sqrt{\frac{T}{2}}\right)$
$\underline{s_3}$: $\left(-\sqrt{\frac{T}{2}}, \sqrt{\frac{T}{2}}\right)$.

All 3 signals are at the same distance (namely $\sqrt{T}$) from the origin indicating that they have the same energy.

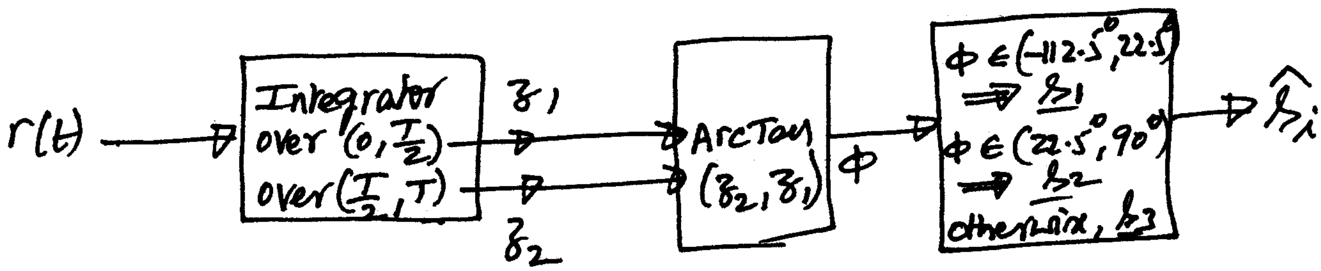
Part (c) Since all 3 signals have same energy and are all equally likely, minimum P_e decision boundaries are obtained by drawing perpendicular bisectors as follows. Since these 3 points are on a circle of radius $\sqrt{T}$ centered at origin, all bisectors go through the origin.





<u>Part (d)</u> First, we need to form $z_i = \int_0^T r(t)\psi_i(t)\,dt \quad i=1,2.$

But since $\psi_1(t)$ and $\psi_2(t)$ are pulses of height $\sqrt{\frac{2}{T}}$ and durations from 0 to $\frac{T}{2}$ and $\frac{T}{2}$ to T, respectively, we need a an integrator that resets to 0 after every $T/2$ seconds.



If $\tan^{-1}(z_2, z_1) = \phi$ is an angle

between -112.5^0 to $+22.5^0$ $\rightarrow$ choose $s_1(t)$

between 22.5^0 to 90^0 $\Rightarrow$ choose $s_2(t)$

between 90^0 to -112.5^0 $\Rightarrow$ choose $s_3(t)$.