Dept. of ECE, CMU



18-550

FALL 1999

SOLUTIONS TO HW#5

1-) Problem 5.3 from Sklar

a) $P_E \triangleq p(\text{symbol error}) = 10^{-3}$

$P_M^U \triangleq p(\text{message error for an uncoded block}) = 1-(1-P_E)^{92}$

$\Rightarrow P_M^U = 1-(1-10^{-3})^{92} = 8.79 \times 10^{-2}$

b) $P_E = 10^{-3}$

$P_M^C \triangleq p(\text{message error after coding})$

Since (127,92) code is capable of triple error corrections, $P_M^C = \sum_{j=4}^{127} \binom{127}{j} P_E^j (1-P_E)^{(127-j)}$

$\cong \binom{127}{4} (10^{-3})^4 (1-0.001)^{123}$

$= 9.13 \times 10^{-6}$

2-) Problem 5.4 from Sklar

$P_E^U = p(\text{Symbol error without coding})$

$P_E^U = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$ for coherent BPSK

$= Q\left(\sqrt{2 \times 10}\right) = Q(4.47) \cong 4.05 \times 10^{-6}$

$$P_M^U = 1-(1-P_E^U)^{12} = 1-(1-4.05\times10^{-6})^{12} = 4.86\times10^{-5}$$

For the (24,12) code, the code rate is $\frac{1}{2}$. Thus, the datarate is double the uncoded rate and the coded signal to noise ratio, $\frac{E_c}{N_0}$, is 3dB less than the $\frac{E_b}{N_0}$.

$$\Rightarrow \frac{E_c}{N_0} = 7\text{ dB} = 10^{0.7} = 5.01$$

$$P_E^C = P(\text{symbol error after coding}) = Q\left(\sqrt{\frac{2E_c}{N_0}}\right)$$
$$= Q(\sqrt{2\times5.01}) = Q(3.16) = 0.0008$$

Since the code is capable of double error corrections

$$P_M^C = \sum_{j=3}^{24} \binom{24}{j} (P_E^C)^j (1-P_E^C)^{(24-j)}$$
$$\cong \binom{24}{3} (8\times10^{-4})^3 (1-8\times10^{-4})^{21} \cong 1.02\times10^{-6}$$

$$\text{Performance improvement using Coding} = \frac{4.86\times10^{-5}}{1.02\times10^{-6}} = 47.6$$

3-) Problem 5.8 from Sklar

a) $\underline{u} = \underline{m}\, G$ → generator matrix.

$\underline{u}$: code vector

$\underline{m}$: message vector

$n = 7$

$k = 4$

$(n-k) = 3$

| message vector | code vector |
|---|---|
| 0 000 | 000 00 00 |
| 0 001 | 11 0 0001 |
| 00 10 | 011 0010 |
| 00 11 | 101 0011 |
| 01 00 | 101 0100 |
| 0 101 | 011 0101 |
| 0 110 | 1100 110 |
| 0 111 | 0000 111 |
| 1 000 | 1111 000 |
| 1 001 | 0011 001 |
| 1 010 | 1001010 |
| 1 011 | 0101011 |
| 1 100 | 0101100 |
| 1 101 | 100 1101 |
| 1 110 | 0 01 1110 |
| 1 111 | 111 1111 |

b) $G = [P \,\vdots\, I_4] \Rightarrow P = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \Rightarrow H = \left[ I_{7-4} \,\vdots\, P^T \right]$

$$\Rightarrow H = \left[\begin{array}{ccc:cccc} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{array}\right]$$

c) $\underline{r} = [1101101]$

$$\underline{S} = \underline{r}\, H^T = [1101101] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} = [0\ 1\ 0]$$

Thus 1101101 is not a valid codeword
( $\underline{S} \neq \underline{0}$ )

d) $d_{min}$ = minimum weight of nonzero code vectors
$= 3$ (See part (a))

error-correcting capability $= t = \left\lfloor \frac{d_{min}-1}{2} \right\rfloor = 1$

$\therefore$ it can correct only single errors

e) It detects upto $d_{min} - 1 = 2$ errors.

4) Problem 5.10 from Sklar

$\underline{U} = \underline{m}\, G$ and $\underline{m} = [m_1\ m_2\ m_3\ m_4\ m_5]$

$\underline{U} = [U_1\ U_2\ U_3\ U_4\ U_5\ U_6\ U_7\ U_8\ U_9]$

if we put the given equations into vector matrix form $\Rightarrow$

$$\underline{U} = \begin{bmatrix} m_1 & m_2 & m_3 & m_4 & m_5 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{= G}$$

b) The parity-check matrix $= H = \left[ I_{n-k} \,\vdots\, P^T \right]$

$$\Rightarrow H = \left[ I_{9-5} \,\vdots\, P^T \right] = \left[\begin{array}{cccc:ccccc} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \end{array}\right]$$

c) $n=9$, $k=5$

$d_{min}$ = minimum weight of 32 code vectors $= 3$

Since all columns of $\underline{H}$ are distinct, adding two columns of $\underline{H}$ won't lead to zero vector.

However, adding 3 column of $\underline{H}$ (e.g., 1st, 5th and 7th) leads to zero vector. $\therefore d_{min} = 3$.

5-) Problem 5.15 from Sklar $n=15$, $K=11$

a) $H = \left[ I_{\underbrace{n-k}_{4}} \,\vdots\, P^T \right] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$

b) $2^4 = 16$ error pattern $\rightarrow$ 16 coset leaders

The list of coset leaders is

0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 → No error pattern
0 0 0 0 0 0 0 0 0 0 0 0 0 0 1
0 0 0 0 0 0 0 0 0 0 0 0 0 1 0
0 0 0 0 0 0 0 0 0 0 0 0 1 0 0
0 0 0 0 0 0 0 0 0 0 0 1 0 0 0
0 0 0 0 0 0 0 0 0 0 1 0 0 0 0
0 0 0 0 0 0 0 0 0 1 0 0 0 0 0
0 0 0 0 0 0 0 0 1 0 0 0 0 0 0
0 0 0 0 0 0 0 1 0 0 0 0 0 0 0
0 0 0 0 0 0 1 0 0 0 0 0 0 0 0
0 0 0 0 0 1 0 0 0 0 0 0 0 0 0
0 0 0 0 1 0 0 0 0 0 0 0 0 0 0
0 0 0 1 0 0 0 0 0 0 0 0 0 0 0
0 0 1 0 0 0 0 0 0 0 0 0 0 0 0
0 1 0 0 0 0 0 0 0 0 0 0 0 0 0
1 0 0 0 0 0 0 0 0 0 0 0 0 0 0

15 single error patterns

it is a perfect code because the 16 coset leaders allow for the correction of all single-error patterns.

c) $\underline{v} = [0\,11111\,0\,0\,1\,0\,11\,0\,11]$

$\underline{s} = \underline{v}\,H^T = [0110]$ This is same as the 8th column of $\underline{H}$ indicating an error in the 8th bit.

The coset leader resulting in the syndrome $[0110]$ is

$\underline{e} = 0000\,0001\,0000000$

The correct codeword $\hat{\underline{u}} = \underline{v} + \underline{e} = [011\,111\,011\,011\,011]$

d) Since it is a perfect code, $d_{min} = 2t+1$ and $t=1$

$\Rightarrow d_{min} = 3$

Therefore, $\rho = d_{min} - 1 = 2$ erasures can be corrected.

For example, if vector $\underline{r} = [\,x\,x\ 1111\ 0110110\ 11\,]$ is received (where xx stands for 2 erasures) it will be decoded as 011111011011011 since $\underline{r}$ is closest in Hamming distance to this codeword than to any of the other codewords. (when comparing the right most 13 digits)

6) Problem 5.17 from Sklar

$n \leq 7$ & if $x^n + 1 \bmod g(x) = 0$, then $g(x)$ is the generator polynomial

a) $g(x) = 1 + x^3 + x^4$, $n-k=4 \Rightarrow n = k+4 \leq 7 \Rightarrow k \leq 3$

Test the possible n,k values

$k=1, n=5$ : $x^5 + 1 \bmod g(x) = \overbrace{x^3 + x}^{\text{obtained through long division}} \neq 0$

it is not a generator for $n=5$

$k=2, n=6$ : $\cancel{x}^6 + 1 \bmod g(x) = x^3 + x^2 + x \neq 0$, it is not a generator

$k=3,\ n=7$: $\quad x^7+1 \bmod g(x) = x^2+x \neq 0$ it is not a generator.

b) $g(x) = 1+x^2+x^4$ $\quad n-k=4 \Rightarrow n=k+4 \leq 7 \Rightarrow k \leq 3$

$k=1,\ n=5$: $\quad x^5+1 \bmod g(x) = x^3+x+1 \neq 0$, No

$k=2,\ n=6$: $\quad x^6+1 \bmod g(x) = 0$ $\quad$ Yes

a cyclic code that can be generated for $n=6, k=2$

$k=3,\ n=7$: $\quad x^7+1 \bmod g(x) = x+1 \neq 0$ No

c) $g(x) = 1+x+x^3+x^4$ $\quad n-k=4 \Rightarrow k \leq 3$

$k=1,\ n=5$: $\quad x^5+1 \bmod g(x) = x^3+x^2 \neq 0$ No

$k=2,\ n=6$: $\quad x^6+1 \bmod g(x) = 0$ $\quad$ Yes

a cyclic code that can be generated for $n=6, k=2$

$k=3,\ n=7$ $\quad x^7+1 \bmod g(x) = x+1 \neq 0$ No

d) $g(x) = 1+x+x^2+x^4$ $\quad n-k=4 \Rightarrow k \leq 3$

$k=1,\ n=5$: $\quad x^5+1 \bmod g(x) = x^3+x^2+x+1 \neq 0$ No

$k=2, n=6$ : $x^6+1 \mod p(x) = x^3+x+1 \neq 0$ NO

$k=3, n=7$: $x^7+1 \mod p(x) = 0$ Yes

a cyclic code that can be generated for $n=7, k=3$

e) $g(x) = 1+x^3+x^5$ $\quad n-k=5 \Rightarrow k \leq 2$

$k=1, n=6$ : $\quad x^6+1 \mod p(x) = x^4+x+1 \neq 0$ NO

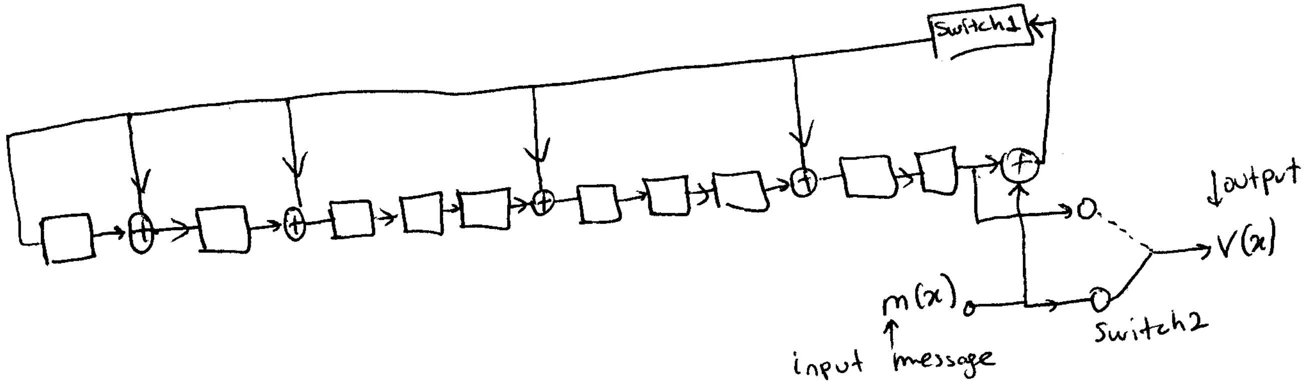
$k=2, n=7$: $\quad x^7+1 \mod p(x) = x^3+x^2 \neq 0$ NO

7-) Problem 5.21 from Sklar

$n=15, k=5$

A (15,5) cyclic code with $g(x) = 1+x+x^2+x^5+x^8+x^{10}$

$n-k=10$, the encoder for this code is




b) $m(x) = 1+x^2+x^4$

$x^{n-k} m(x) = x^{10}(1+x^2+x^4) = x^{10}+x^{12}+x^{14}$

$= \underbrace{(1+x+x^2+x^5+x^8+x^{10})}_{g(x)}(x^4+1) + \underbrace{x^9+x^8+x^6+x^4+x^2+x+1}_{r(x)}$

$\Rightarrow u(x) = r(x) + x^{n-k} m(x)$

codeword $= 1 + x + x^2 + x^4 + x^6 + x^8 + x^9 + x^{10} + x^{12} + x^{14}$

$\Rightarrow \underline{u} = [\underbrace{111\ 01\ 01\ 011}_{\text{parity bits}}\ \underbrace{10101}_{\text{message bits.}}]$

c) $v(x) = 1 + x^4 + x^6 + x^8 + x^{14} = g(x)(1 + x^2 + x^4) + \underbrace{x + x^3 + x^4 + x^7 + x^9}_{r(x)}$

since $r(x) \neq 0$, $v(x)$ is not a codeword.

8.) Problem 5.23 from Sklar

The (15,11) code introduces less redundancy so it has less error correcting capability.

The (15,11) code, because of lower redundancy requires less bandwidth. The trade-off is required power versus required bandwidth.

g-) The printouts of the uncoded system and coded system are in Fig 1 and Fig 2 respectively.

For all blocks, I've chosen sample time 1 sec. and for all simulations, maximum step size and initial step size are set to be 1.

For uncoded system, simulation stop time is set to be 110 000.

For coded system, simulation stop time is set to be 110022 because coding and decoding add 22 seconds delay between input and decoded sequence.

I obtained BER versus p and CBER versus p graphas in Fig 3 for $p = 0.0001, 0.001, 0.01, 0.02, 0.05, 0.1, 0.2$

As you can see from that graph, coding makes the BER worse after $p = 0.05$. That means that coding does not help to improve BER always.

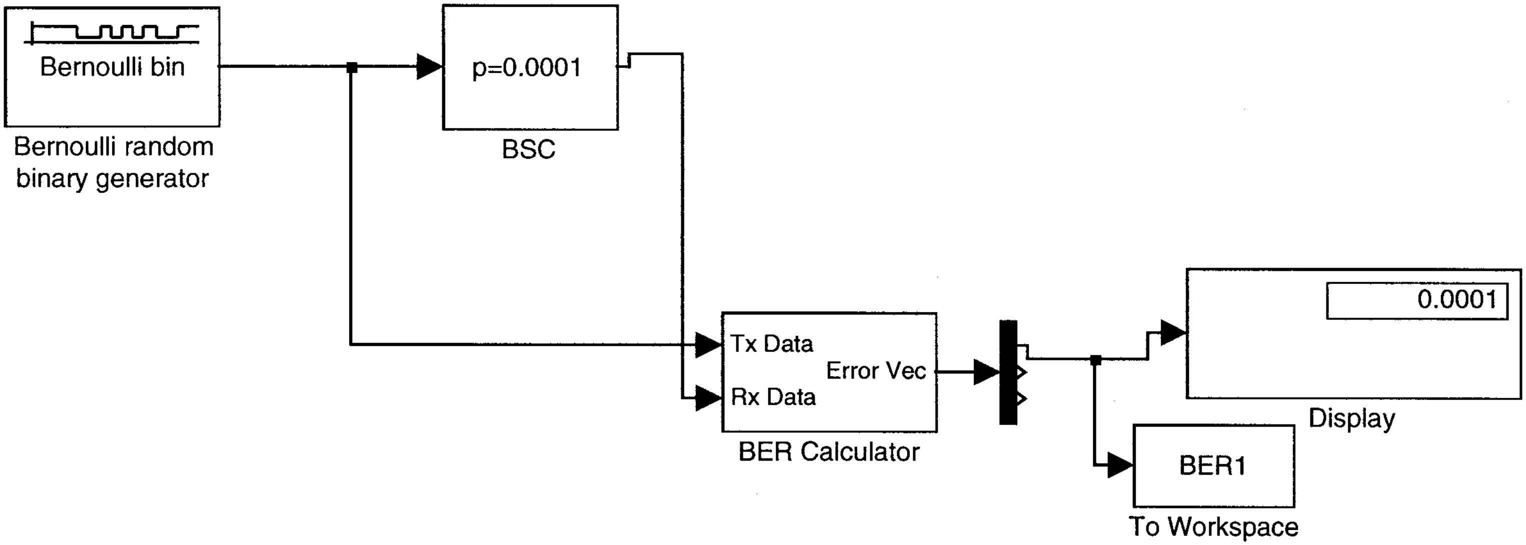

Bernoulli bin
Bernoulli random binary generator
p=0.0001
BSC
Tx Data
Rx Data
Error Vec
BER Calculator
0.0001
Display
BER1
To Workspace

Fig1

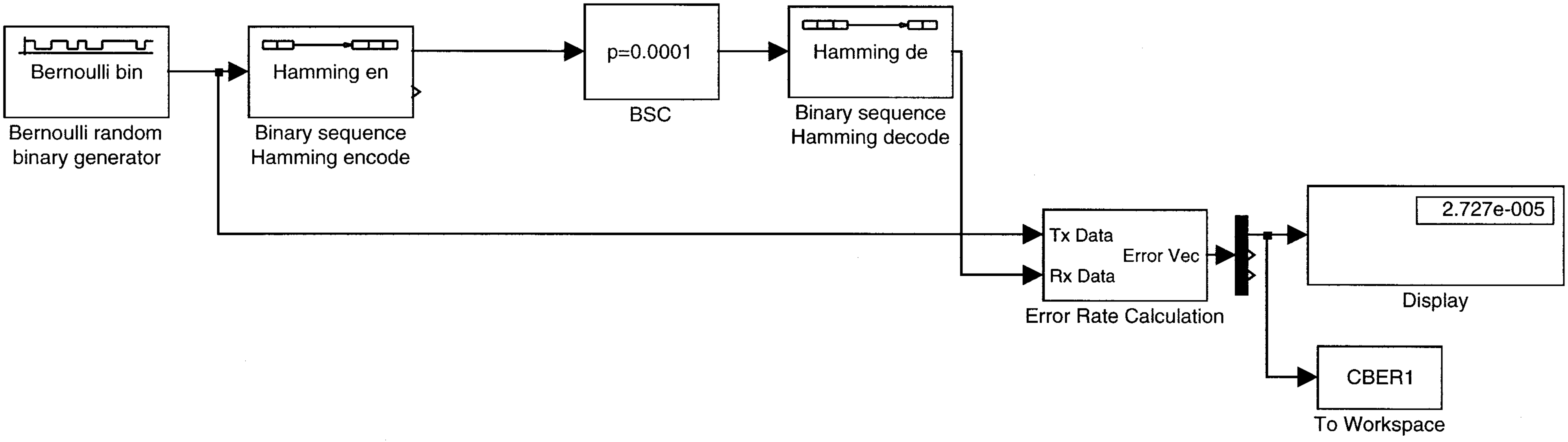
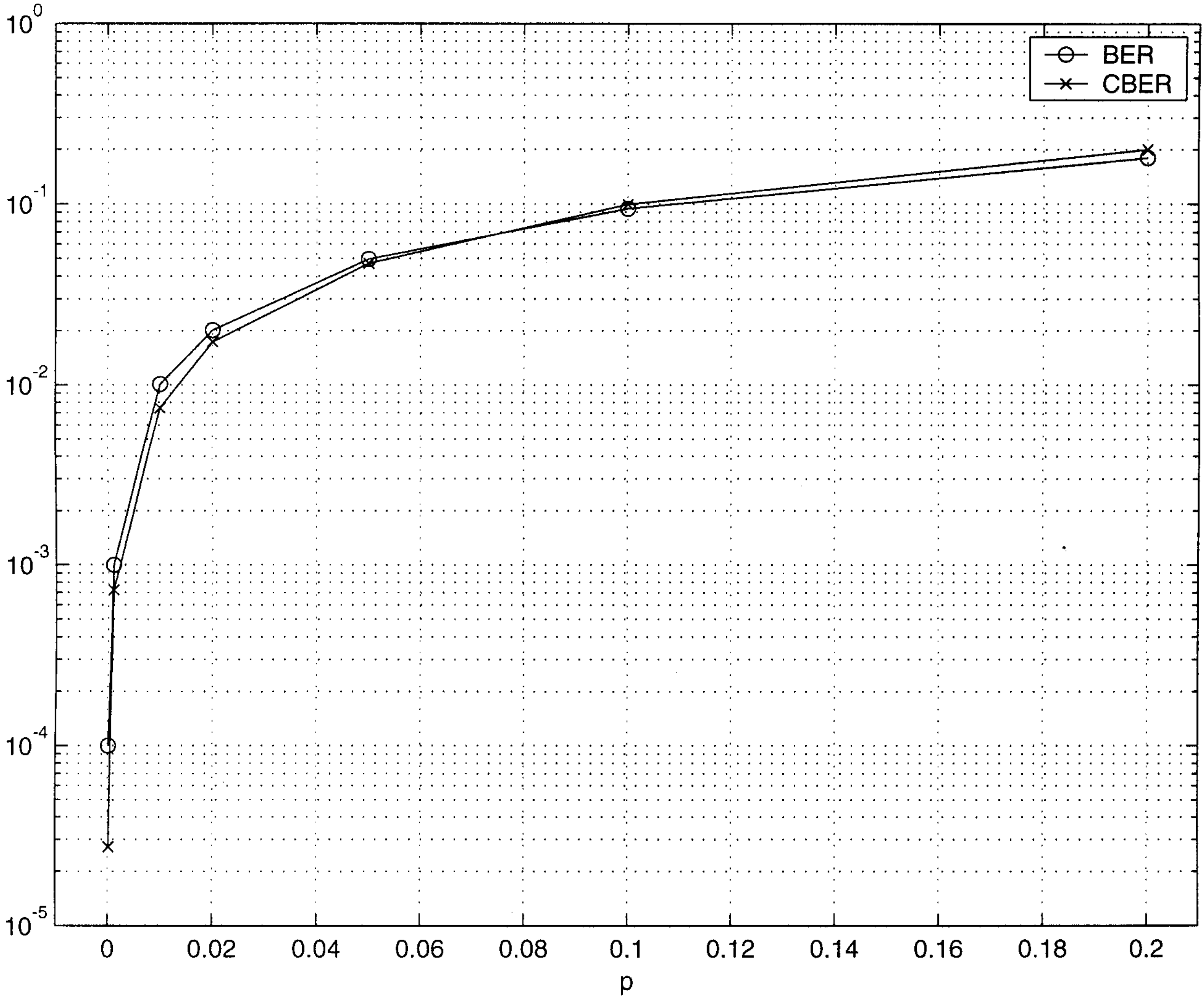

Bernoulli bin
Bernoulli random
binary generator
Hamming en
Binary sequence
Hamming encode
p=0.0001
BSC
Hamming de
Binary sequence
Hamming decode
Tx Data
Rx Data
Error Vec
Error Rate Calculation
2.727e-005
Display
CBER1
To Workspace

Fig 2



BER
CBER
10^0
10^-1
10^-2
10^-3
10^-4
10^-5
0
0.02
0.04
0.06
0.08
0.1
0.12
0.14
0.16
0.18
0.2
p


Fig 3