18-550 HW#4 Solutions

Problem 1 (3.2 in Sklar)

a) Using equation 3.2,

$$\int_{-2}^{2} \varphi_1(t)\,\varphi_2(t)\,dt = \int_{-2}^{-1}(-A)(-A)\,dt + \int_{-1}^{0}(A)(-A)\,dt + \int_{0}^{1}(A)(A)\,dt + \int_{1}^{2}(-A)(A)\,dt$$
$$= A^2 - A^2 + A^2 - A^2 = 0 \quad \checkmark$$

$$\int_{-2}^{2} \varphi_1(t)\,\varphi_3(t)\,dt = \int_{-2}^{-1}(-A)(-A)\,dt + \int_{-1}^{0}(A)(-A)\,dt + \int_{0}^{1}(A)(-A)\,dt + \int_{1}^{2}(-A)(-A)\,dt$$
$$= A^2 - A^2 - A^2 + A^2 = 0 \quad \checkmark$$

$$\int_{-2}^{2} \varphi_2(t)\,\varphi_3(t)\,dt = \int_{-2}^{0}(-A)(-A)\,dt + \int_{0}^{2}(A)(-A)\,dt$$
$$= 2A^2 - 2A^2 = 0 \quad \checkmark$$

So $\varphi_1(t)$, $\varphi_2(t)$ and $\varphi_3(t)$ are pairwise orthogonal over $(-2,2)$.

b) Using equation 3.3, we want each $K_j = 1$. So,

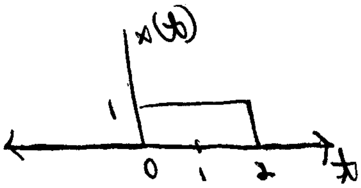
$$\int_{-2}^{2} \varphi_1^2(t)\,dt = \int_{-2}^{-1}(-A)^2\,dt + \int_{-1}^{0}(A)^2\,dt + \int_{0}^{1}(A)^2\,dt + \int_{1}^{2}(-A)^2\,dt = A^2 + A^2 + A^2 + A^2 = 4A^2$$

$$\int_{-2}^{2} \varphi_2^2(t)\,dt = \int_{-2}^{0}(-A)^2\,dt + \int_{0}^{2}(A)^2\,dt = 2A^2 + 2A^2 = 4A^2$$

$$\int_{-2}^{2} \varphi_3^2(t)\,dt = \int_{-2}^{2}(-A)^2\,dt = 4A^2$$

Set $4A^2 = 1 \Rightarrow A^2 = \frac{1}{4}$, so $\boxed{A = \frac{1}{2}}$.

c)



From equations 3.4 and 3.5,

$$x(t) = \sum_{j=1}^{3} a_j \varphi_j(t)$$

Since $K_j = 1$,

$$a_1 = \int_0^1 (1)(\tfrac{1}{2})\,dt + \int_1^2 (1)(-\tfrac{1}{2})\,dt = \tfrac{1}{2} - \tfrac{1}{2} = 0$$

$$a_2 = \int_0^2 (1)(\tfrac{1}{2})\,dt = 1$$

$$a_3 = \int_0^2 (1)(-\tfrac{1}{2})\,dt = -1$$

So $\boxed{x(t) = \varphi_2(t) - \varphi_3(t)}$, where $A = \frac{1}{2}$ in $\varphi_j(t)$.

See also Example 3.1.

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Problem 2 (3.4 in Sklar)

coherent BPSK system, rate = 5000 bps, $A = 1\,mV$, $N_0 = 10^{-11}$ W/Hz.

$s_1(t) = A\cos w_0 t \qquad s_2(t) = -A\cos w_0 t.$

From equation 3.26,

Let $A = \sqrt{\frac{2E}{T}}$, where E is the energy/bit and T is the bit duration.

$\exists\ E = \frac{A^2 T}{2} = \frac{(0.001)^2\left(\frac{1}{5000}\right)}{2} = 1\times10^{-10}$ Joules

From equation 3.84,

$P_B = Q\left(\sqrt{\frac{2E_b}{N_0}}\right) = Q\left(\sqrt{\frac{2\times10^{-10}}{10^{-11}}}\right) = Q(\sqrt{20}) \approx Q(4.47).$

For $Q(x)$ when $x>3$,

$P_B = \frac{1}{\sqrt{20}\cdot\sqrt{2\pi}}\, e^{-\frac{(\sqrt{20})^2}{2}} = 4.05\times10^{-6}$

So, the expected number of bit errors in a day is

$\frac{5000 \text{ bits}}{\text{sec}} \times \frac{86400 \text{ sec}}{\text{day}} \times \frac{4.05\times10^{-6} \text{ errors}}{\text{bit}} \approx \boxed{1750 \text{ bit errors / day.}}$

See also Example 3.4.

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Problem 3 (3.7 in Sklar)

For binary noncoherent orthogonal FSK,

From equation 3.111: $P_B = \frac{1}{2} e^{-\frac{E_b}{2N_0}} = \frac{1}{2} e^{-\frac{1}{2}(10^{1.3})} \cong 2.32 \times 10^{-5}$

For binary coherent PSK,

From equation 3.84: $P_B = Q\left(\sqrt{\frac{2E_b}{N_0}}\right) = Q\left(\sqrt{2 \cdot (10^{.8})}\right) \cong Q(3.55)$

$$\approx \frac{1}{3.55\sqrt{2\pi}} e^{-\frac{3.55^2}{2}} = 2.06 \times 10^{-4}$$

So, choose the FSK system.

Problem 4 (3.8 in Sklar)

Using Figure 3.16a and equations 3.61 and 3.62,

original message:		1	0	1	0	1	0	1	1	1	1	0	1	0	1	0	1	0	0	0	0	1	1	1	1
eqtn 3.61, initial bit=0	0	1	1	0	0	1	1	0	1	0	1	1	0	0	1	1	0	0	0	0	0	1	0	1	0
eqtn 3.61, initial bit=1	1	0	0	1	1	0	0	1	0	1	0	0	1	1	0	0	1	1	1	1	1	0	1	0	1
eqtn 3.62, initial bit=0	0	0	1	1	0	0	1	1	1	1	1	0	0	1	1	0	0	1	0	1	0	0	0	0	0
eqtn 3.62, initial bit=1	1	1	0	0	1	1	0	0	0	0	0	1	1	0	0	1	1	0	1	0	1	1	1	1	1

To generate the first two sequences, Equation 3.61 was used with initial bits 0 and 1. $c(k) = c(k-1) \oplus m(k)$ means that encoded bit k equals the result of XOR'ing the previous encoded bit and the current message bit.

The second set of sequences uses equation 3.62, where the XOR result is inverted to produce the current encoded bit.

Because the current encoded bit is dependent on the previous encoded bit, an arbitrary initial bit is needed to produce the first encoded bit.

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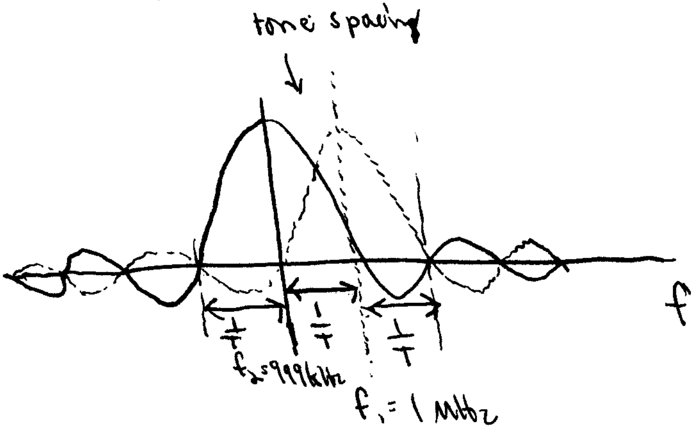
Problem 5 (3.9 in Sklar)

a) From Section 3.6.4 and Example 3.3,

BFSK is orthogonal when two tones separated by $\frac{1}{T} = \frac{1}{10^{-3}} = 1\text{kHz}$.

So, lower tone is at 999kHz.



total bandwidth $= \frac{2}{T} + \frac{1}{T} = \frac{3}{T} = \boxed{3\text{ KHz}}$

b) For MFSK,

there are still 2 $\frac{1}{T}$ portions of the bandwidth due to the highest and lowest frequency signals, and M-1 $\frac{1}{T}$-width tone spacings in-between. So,

total bandwidth $= \frac{2}{T} + M-1\left(\frac{1}{T}\right) = \frac{M+1}{T} = \boxed{(M+1)\text{ kHz}}$

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Problem 6 (3.10 in Sklar)

For BPSK, from equation 3.93: $P_B = Q\left(\sqrt{\frac{E_b(1-\rho)}{N_0}}\right)$

From Table B.1, $P_B = 2\times10^{-3} \Rightarrow x = 2.88$

$P_B = 2.5\times10^{-3} \Rightarrow x = 2.81$

So,

$$2.88 = \sqrt{\frac{E_b(1-\rho_1)}{N_0}} \quad \text{and} \quad 2.81 = \sqrt{\frac{E_b(1-\rho_2)}{N_0}}$$

$\rho_1 = -1$ for no phase error, solve for ρ_2.

$$2.88 = \sqrt{\frac{2E_b}{N_0}}$$

$$\Rightarrow \frac{E_b}{N_0} = \frac{2.88^2}{2} = 4.1472$$

$$2.81 = \sqrt{(4.1472)(1-\rho_2)}$$

$$7.8961 = 4.1472(1-\rho_2)$$

$$\Rightarrow \rho_2 = -0.904$$

From equation 3.91,

$\rho_2 = \cos\theta_2 \Rightarrow \theta_2 = \cos^{-1}\rho_2 = 154.69$ degrees

$\rho_1 = \cos\theta_1 \Rightarrow \theta_1 = \cos^{-1}\rho_1 = 180$ degrees

phase error $\phi = 180 - 154.69 = \boxed{25.31 \text{ degrees}}$

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Problem 7 (3.11 in Sklar)

General form of P_B for coherent binary signaling: $P_B = Q\left(\sqrt{\frac{E_b(1-\rho)}{N_0}}\right)$ (equation 3.93)

$E_b = \frac{A^2 T}{2} = \frac{0.5^2 (0.01)}{2} = 1.25 \times 10^{-3}$ Joules $\quad \frac{N_0}{2} = 1 \times 10^{-4} \Rightarrow N_0 = 2 \times 10^{-4}$

From equation 3.89,

$$\rho = \frac{1}{E_b}\int_0^T s_1(t) s_2(t)\, dt = \frac{1}{1.25 \times 10^{-3}} \int_0^{0.01} 0.5 \cos(2\pi \, 1000 t) \cdot 0.5 \cos(2\pi \, 1010 t)\, dt$$

$$= \frac{0.25}{1.25 \times 10^{-3}} \int_0^{0.01} \frac{1}{2}\cos(2\pi \cdot 2010 t) + \frac{1}{2}\cos(2\pi(-10 t))\, dt$$

$$= \frac{1}{2}(200) \int_0^{0.01} \cos(2\pi \cdot 2010 t) + \cos(2\pi \cdot 10 t)\, dt$$

$$= 100 \left[\frac{1}{4020\pi}\sin(2\pi \cdot 2010 t) + \frac{1}{20\pi}\sin(2\pi \cdot 10 t)\right]_0^{0.01}$$

$$= 0.94$$

So,

$$P_B = Q\left(\sqrt{\frac{1.25 \times 10^{-3}(1-0.94)}{2 \times 10^{-4}}}\right) = Q(0.61) = \boxed{0.2709}$$

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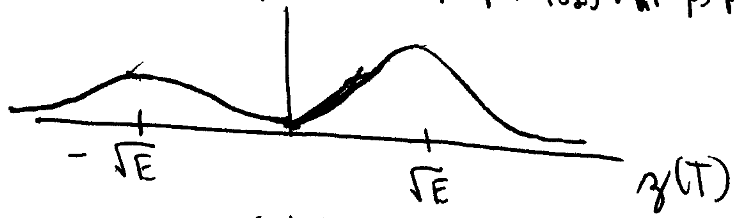
Problem 8 (3.14 in Sklar)

a) Since s_1 and s_2 are equally likely, $\gamma = 0$. However, since the switch is faulty with probability p, $p(z|s_1)$ and $p(z|s_2)$ look like (for $p < 0.5$):

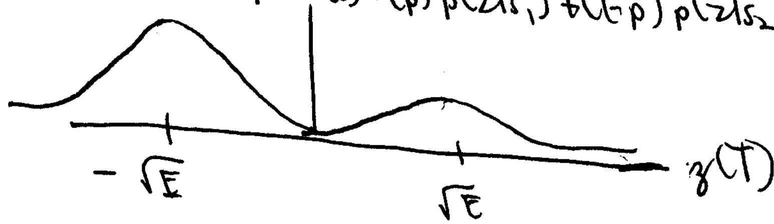
$$p(z|s_1) = (p)\, p(z|s_2) + (1-p)\, p(z|s_1)$$



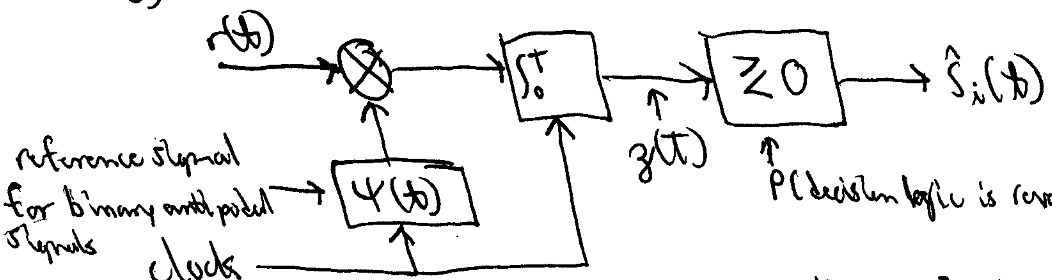
$p(z|s_1)$: Gaussian with mean $\sqrt{E}$, variance $\frac{N_0}{2}$

$p(z|s_2)$: Gaussian with mean $-\sqrt{E}$, variance $\frac{N_0}{2}$

$$p(z|s_2) = (p)\, p(z|s_1) + (1-p)\, p(z|s_2)$$



b)



P(decision logic is reversed for $p > 0.5$)

For $p < \frac{1}{2}$, this does not change since $p(z|s_1) > p(z|s_2)$ for $z > 0$ and $p(z|s_1) < p(z|s_2)$ for $z < 0$.

See Figure 3.7b with $N=1$
See also equations 3.41–3.42c

c) $p = 0.1$, $\frac{E_b}{N_0} = \infty$. With infinite SNR, $Q(\infty) = 0$. However, since the switch is faulty, the detector will still choose the wrong signal with probability 0.1.

$p = 0$, $\frac{E_b}{N_0} = 7\,\text{dB}$. For antipodal signals, $P_B = Q\left(\sqrt{2 \cdot (10^{.7})}\right) = Q(3.17) = 8 \times 10^{-4}$.

So, choose the <u>$p = 0$ system</u>.