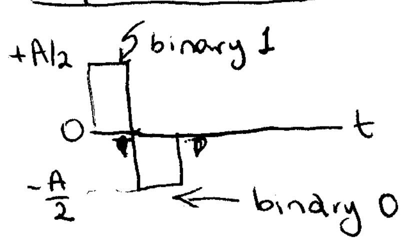
DEPT. OF ECE, CMU

18-550 Fundamental of Communication Systems HW-3 Solution

1-) Problem 2.13 from Sklar

Let the peak to peak amplitude separation be A volts.

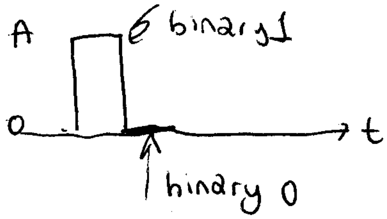
Bipolar Case:



$$\text{Average power} = \left(\frac{A}{2}\right)^2 \Pr\left(\frac{A}{2}\right) + \left(-\frac{A}{2}\right)^2 \Pr\left(-\frac{A}{2}\right)$$

$$= \frac{A^2}{4}\cdot\frac{1}{2} + \frac{A^2}{4}\cdot\frac{1}{2} = \frac{A^2}{4}$$

Unipolar Case



$$\text{Average power} = A^2 \Pr(A) + 0^2 \Pr(0)$$

$$= \frac{A^2}{2}$$

$\therefore$ Bipolar signal uses less average power than unipolar signal.

Bipolar is 3 dB better than unipolar.

2-) Problem 2.16 from Sklar

a) $P(s_1) = P(s_2) = 0.5$

Using Equation (2.30)

$$\frac{p(z|s_1)}{p(z|s_2)} \underset{H_2}{\overset{H_1}{\gtrless}} \frac{P(s_2)}{P(s_1)}$$

with equally-likely probabilities, the optimum threshold from equation (2.31) becomes

$$z(T) \underset{H_2}{\overset{H_1}{\gtrless}} \frac{a_1 + a_2}{2} = \gamma_0$$

and $a_1 = \int_0^T 1\, dt = T$ and $a_2 = \int_0^T (-1) dt = -T$

$$\Rightarrow \boxed{\gamma_0 = \frac{T - T}{2} = 0}$$

b) $P(s_1) = 0.7$ then $P(s_2) = 0.3$

Using Equation (B.12)

$$\frac{z(a_1 - a_2)}{\sigma_0^2} - \cancel{\frac{a_1^2 - a_2^2}{2\sigma_0^2}}^{\,0} \underset{H_2}{\overset{H_1}{\gtrless}} \ln\left(\frac{P(s_2)}{P(s_1)}\right)$$

$$z \underset{H_2}{\overset{H_1}{\gtrless}} \frac{\sigma_0^2}{a_1 - a_2} \ln \frac{P(s_2)}{P(s_1)} = \gamma_0$$

$$\Rightarrow \gamma_0 = \frac{0.1}{2T} \ln\left(\frac{0.3}{0.7}\right) = \frac{-0.043}{T} \text{ Volt}$$

c) similarly

$$\gamma_0 = \frac{0.1}{2T} \ln\left(\frac{0.8}{0.2}\right) \quad \text{since } P(s_2) = 1 - 0.2 = 0.8$$

$$\gamma_0 = \frac{0.069}{T} \text{ volt}$$

d) The apriori probabilities have the effect of positioning γ_0 so as to yield a greater probility of correct decisions. For example, when $p(s_1)$ is reduced to 0.2 from 0.5, γ_0 in the Figure 2.25, moves to the right so that samples at the tail of the $p(z|s_2)$ have a greater chance of being declared members of the signal class S_2.

3-) Problem 2.19 from Sklar

a) $M = 16$ levels $= 2^k \Rightarrow k = 4$ bits/symbol

$$R_s = \frac{R}{\log_2 M} = \frac{10 \text{ M bits/s}}{4 \text{ bits/symbol}} = 2.5 \text{ M symbols/sec}$$

$$\text{min BW} = \frac{R_s}{2} = 1.25 \text{ MHz}$$

b) $W = \frac{1}{2}(1+r) R_s = 1.375 \text{ MHz}$

$\Rightarrow (1+r) \cdot (1.25 \text{ MHz}) = (1.375 \text{ MHz})$

$\Rightarrow 1+r = 1.1 \Rightarrow \boxed{r = 0.1}$

4-) Problem 2.21 from Sklar

a) $$R_s = \frac{9600 \text{ bits/s}}{\log_2(8) \text{ bits/smbol}} = 3200 \text{ symbols/sec}$$

b) $W = \frac{1}{2}(1+r) R_s = 2.4 \text{ kHz}$

$\Rightarrow (1+r) \cdot \frac{(3200)}{2} \text{Hz} = (2.4 \text{ kHz}) \Rightarrow 1+r = \frac{2.4}{1.6} = 1.5$

$\Rightarrow \boxed{r = 0.5}$

5) Problem 2.24 from Sklar

$$W = 100 \text{ kHz}, \quad L = 32, \quad r = 0.6$$

a) $W = \frac{1}{2}(1+r) R_s$

$100 \text{ k Hz} = \frac{(1.6)}{2} R_s \Rightarrow R_s = 125 \text{ k symbols/sec}$

For binary PCM, $R = R_s = 125 \text{ k bits/sec}$

(b) $L = 32 = 2^{\ell} \Rightarrow \ell = 5$ bits/sample

$R = \ell \cdot f_s \Rightarrow f_s = \frac{125 \text{ k bits/sec}}{5 \text{ bits/sample}} = 25 \text{ k samples/sec}$

$(W_{analog})_{max} = \frac{f_s}{2} = \underline{\underline{12.5 \text{ kHz}}}$

c) $M = 8 \text{ level} = 2^k \Rightarrow k = 3$ bits/symbol

$\Rightarrow R = R_s \cdot k = 125 \text{ k symbols/sec} \cdot 3 \text{ bits/symbol}$
$= \underline{\underline{375 \text{ kbits/sec}}}$

and $f_s = \frac{375 \text{k bits/sec}}{5 \text{ bits/sample}} = 75 \text{ k samples/sec}$

$(W_{analog})_{max} = \frac{f_s}{2} = 37.5 \underline{\underline{\text{ k Hz}}}$

6-) $$h(t) = \frac{T_b}{\pi\, PW50} \frac{1}{1+\left(\frac{2t}{PW50}\right)^2}$$

$$h_k \doteq h(t=kb) = \frac{T_b}{\pi\, PW50} \frac{1}{1+\left(\frac{2kT_b}{PW50}\right)^2} = \frac{1}{\pi S} \frac{1}{1+\left(\frac{2k}{S}\right)^2}$$

where $S = \frac{PW50}{T_b}$

a) For $S = 1.0$
I plotted the normalized h_k in Figure 1
and I calculated the fraction of ISI
energy as 0.0521

b) I repeated the same thing for $S=2$ and $S=3$.
I plotted the normalized h_k in Figure 2 and
Figure 3 for the normalized density of $S=2$
and $S=3$ respectively.

As you can see from Figures [1-3], when
we increase the data density, ISI increases.
For $S=2$, the fraction of ISI energy

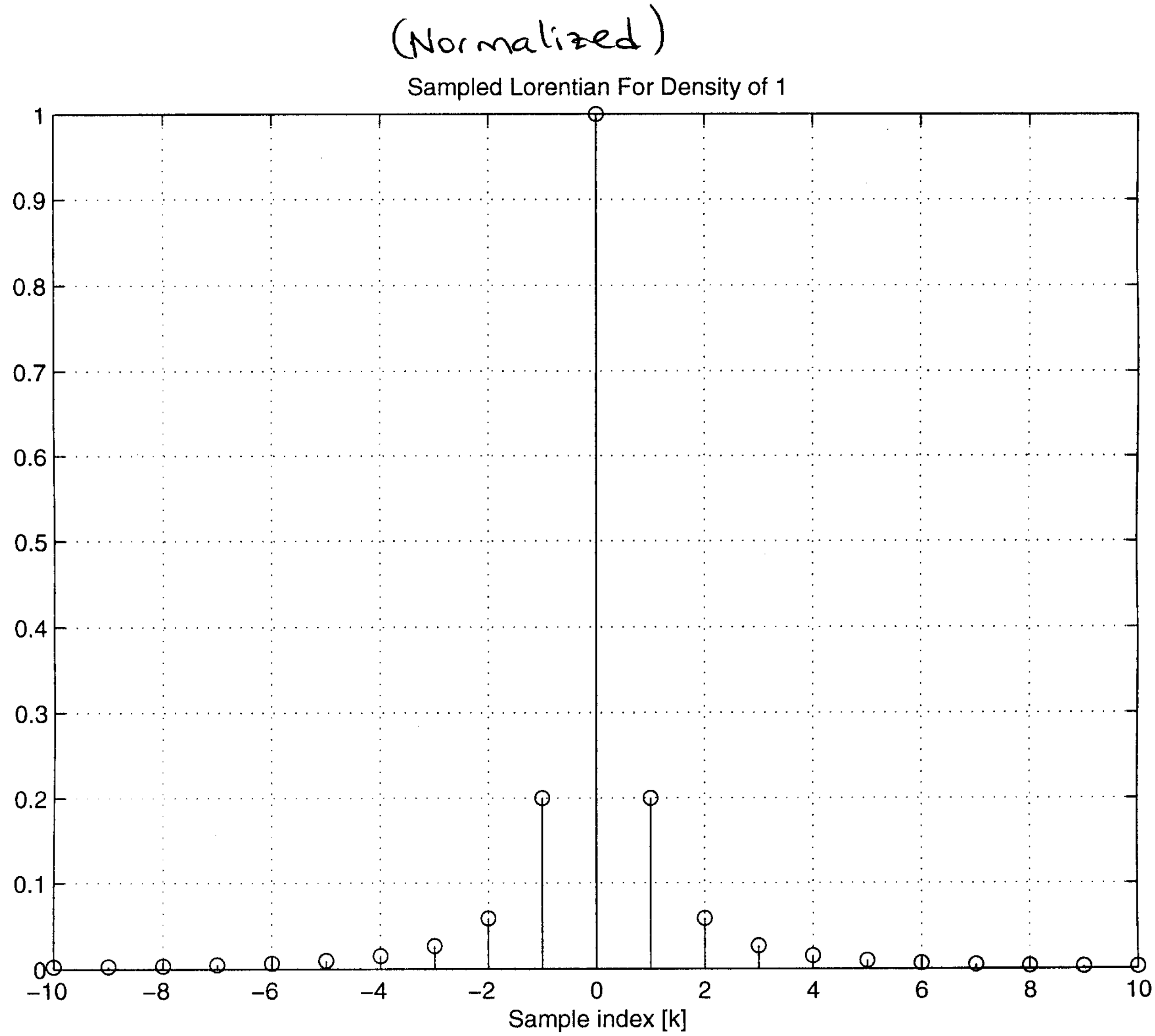

(Normalized)
Sampled Lorentian For Density of 1
Sample index [k]
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
−10
−8
−6
−4
−2
0
2
4
6
8
10

Figure 1

(Normalized)

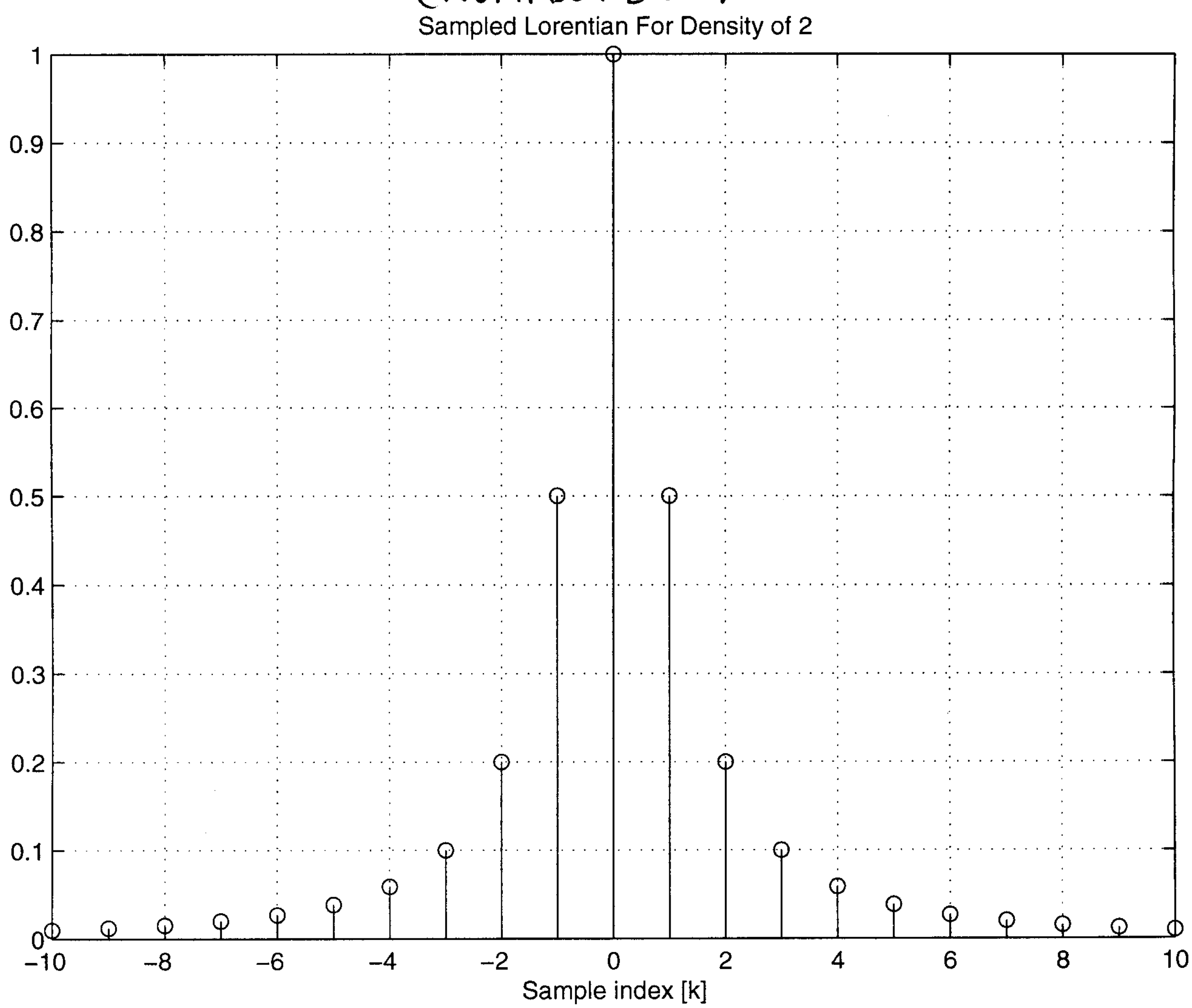


Figure 2

(Normalized)

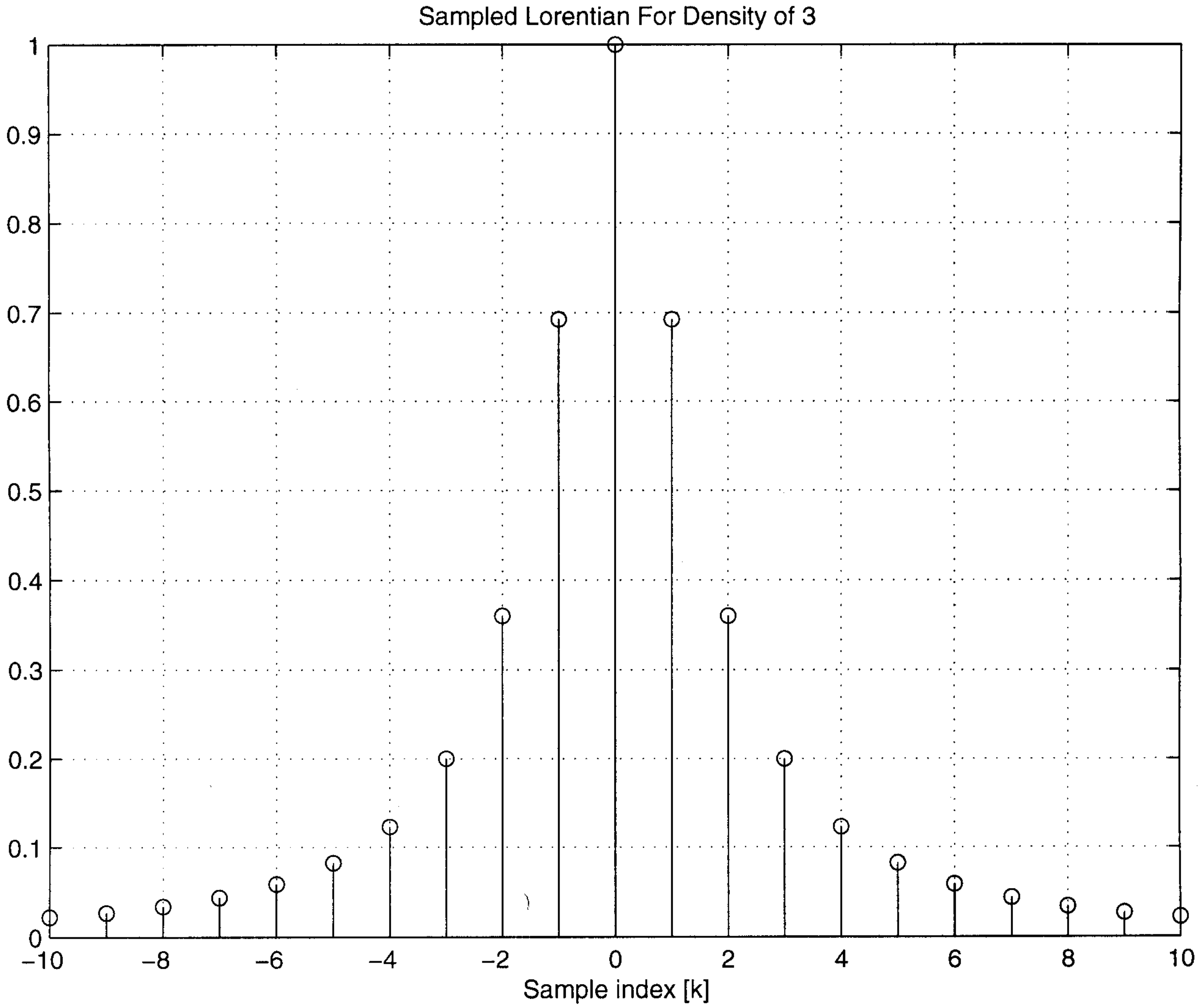


Figure 3

energy is 0.1944 and it is 0.2880 for $s=3$. As expected, increase in the density causes increase in the fraction of ISI energy.

c) From Appendix A pg 731, we know the Fourier transform pairs,

$$e^{-a|t|} \overset{\mathcal{F}}{\longleftrightarrow} \frac{2a}{a^2 + (2\pi f)^2}$$

by scaling property of Fourier transform

$$e^{-a|t|\pi} \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{\pi} \frac{2a}{a^2+(2f)^2}$$

multiply both sides by $\frac{a\pi}{2}$

$$\Rightarrow \frac{a\pi}{2} e^{-a|t|\pi} \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{1+\left(\frac{2f}{a}\right)^2}$$

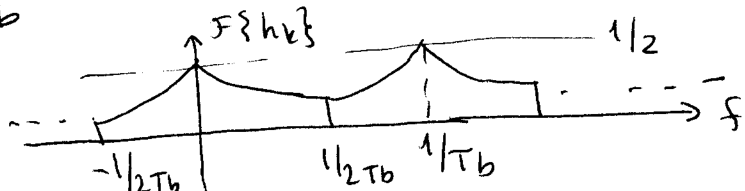
by using the duality Property

$$\frac{1}{1+\left(\frac{2t}{a}\right)^2} \overset{\mathcal{F}}{\longleftrightarrow} \frac{a\pi}{2} e^{-a|f|\pi}$$

So $$\mathcal{F}\{h(t)\} = H(f) = \frac{T_b}{(\pi \cancel{PW50})} \cdot \frac{\cancel{PW50\,\pi}}{2} e^{-PW50|f|\pi}$$

$$H(f) = \frac{T_b}{2} e^{-p_{WSO}|f|\pi}$$

Before sampling, we need to have ideal low-pass filter with a cut-off frequency of $\frac{1}{2T_b}$. Then fourier transform of h_k is $\frac{H(f)}{T_b}$ for $f \in \left[-\frac{1}{2T_b}, \frac{1}{2T_b}\right]$ and periodic with $\frac{1}{T_b}$, which is pictured as below.

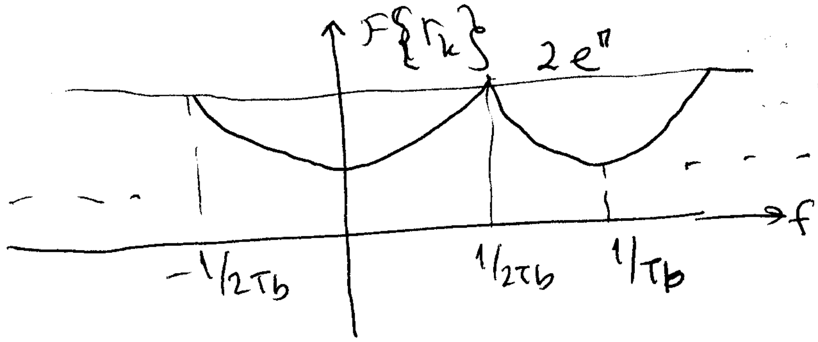


unit rectangular fcn.

Then $$F\{r_k\} = \sum_{k=-\infty}^{\infty} \frac{\Pi\left(\frac{f}{T_b} + \frac{k}{T_b}\right)}{H\left(f + \frac{k}{T_b}\right)}$$

$$= 2 \sum_{k=-\infty}^{\infty} \Pi\left(\frac{f}{T_b} + \frac{k}{T_b}\right) e^{p_{WSO}|f|\pi}$$

(Growing exponential)



Let power spectral density of the white noise be $\frac{N_0}{2}$

Before equalization:

$$\langle \text{noise power} \rangle = \frac{N_0}{2} \frac{1}{T_b}$$

↗ average

After equalization

$$\langle \text{Noise power} \rangle = \frac{N_0}{2} \int_{-1/2T_b}^{1/2T_b} \left(2\, e^{PW50|f|\pi}\right)^2 df$$

$$= \frac{N_0}{2} \cdot (4)\cdot(2) \int_0^{1/2T_b} e^{2PW50 f \pi}\, df$$

$$= \frac{N_0}{2} \cdot (8) \cdot \frac{\left(e^{\frac{2PW50}{2T_b}\pi} - 1\right)}{(2\, PW50\, \pi)} \quad \text{and} \quad \frac{PW50}{T_b} = 2$$

$$= \frac{N_0}{2} \cdot \frac{4}{(PW50)(\pi)} \left(e^{2\pi} - 1\right)$$

Noise power is increased by a factor of

$$x = \frac{\frac{\cancel{N_0}}{\cancel{2}} \frac{4}{(PW50)\pi}\left(e^{2\pi}-1\right)}{\frac{\cancel{N_0}}{\cancel{2}} \frac{1}{T_b}} = \frac{(4)T_b}{(PW50)\pi}\left(e^{2\pi}-1\right)$$

$$\Rightarrow x = \frac{4}{2\pi}\left(e^{2\pi}-1\right) = \frac{2}{\pi}\left(e^{2\pi}-1\right) = \underline{\underline{340.26}}$$