DEPT. OF ECE, CMU

18-550 Fundamental of Communication Systems HW-1 Solution

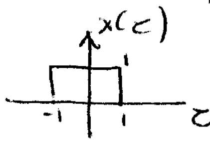
1-) Problem 1.6 from Sklar

Recall that an auto-correlation function, $R(\tau)$, satisfies the following 3 properties

1-) $R(\tau) = R(-\tau)$ 2-) $R(0) \geq |R(\tau)|$ 3-) $F\{R(\tau)\} \geq 0$

Based on those facts, let's look at given

a) $x(\tau) = \begin{cases} 1 & \text{for } |\tau| \leq 1 \\ 0 & \text{otherwise} \end{cases}$



Obviously, $x(\tau)$ satisfies properties number (1) and (2) but third property is violated because the FT of a rectangle is a Sinc(.) function that can be negative.

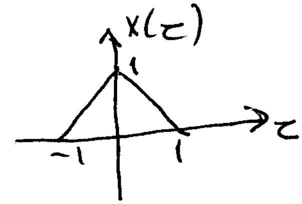
b) $x(\tau) = \delta(\tau) + \sin 2\pi f_0 \tau$

Since Sin (.) is an odd function, $x(\tau) \neq x(-\tau)$ and it is not a valid autocorrelation.

c) $x(\tau) = \exp(|\tau|)$

since $x(\tau)$ actually is minimum at $\tau = 0$, it violates the second property and it is not a valid autocorrelation function.

d-) $x(\tau) = \begin{cases} -\tau+1 & \text{for } 0 \le \tau \le 1 \\ \tau+1 & \text{for } -1 \le \tau \le 0 \end{cases}$



Obviously, $x(\tau)$ satisfies properties number (1) and (2) and $\mathcal{F}\{x(\tau)\} = \operatorname{sinc}^2(f) \ge 0$ for all f. Hence, it is a valid autocorrelation function.

2-) Problem 1.11 from Sklar

$x(t) = A\cos(2\pi f_0 t + \phi)$ where A and f_0 are constants and ϕ is a random variable that is uniformly distributed over $(0, 2\pi)$.

a) $\langle x(t) \rangle = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} A\cos(2\pi f_0 t + \phi)\,dt$

time-average

$= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} A\cos(2\pi f_0 t + \phi)\,dt = 0$

because $x(t)$ is periodic in t with period $T_0 = \frac{1}{f_0}$, and as $T \to \infty$, we can assume that the interval can be approximated as an integer number of periods.

Similarly, $\langle x^2(t) \rangle = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} A^2 \cos^2(2\pi f_0 t + \phi)\, dt$

after integration

$$= \frac{A^2}{2T_0} \left\{ \int_{-T_0/2}^{T_0/2} \underbrace{\cos(4\pi f_0 t + 2\phi)}_{0} + 1 \, dt \right.$$

$$= \frac{A^2}{2T_0} \cdot T_0 = \frac{A^2}{2}$$

b) $m_x = E\{x\} = \int_{-\infty}^{\infty} x(\phi)\, p(\phi)\, d\phi$

where $p(\phi) = \begin{cases} \frac{1}{2\pi} & \text{for } 0 \le \phi \le 2\pi \\ 0 & \text{otherwise} \end{cases}$ Since ϕ is Uniformly distributed over $(0, 2\pi)$

$$E\{x(t)\} = \int_0^{2\pi} \frac{A}{2\pi} \cos(2\pi f_0 t + \phi)\, d\phi = 0 \quad \text{for all } t$$

$$E\{x^2(t)\} = \int_0^{2\pi} A^2 \cos^2(2\pi f_0 t + \phi) \frac{1}{2\pi}\, d\phi$$

$$= \frac{A^2}{2\pi} \frac{1}{2} \int_0^{2\pi} \cos(4\pi f_0 t + 2\phi) + 1 \, d\phi$$

$$= \frac{A^2}{2\pi} \cdot \frac{1}{2} \cdot 2\pi = \frac{A^2}{2} \quad \text{for all } t$$

So $\langle x(t) \rangle = E\{x(t)\}$

$\langle x^2(t) \rangle = E\{x^2(t)\}$

$\therefore$ Process is ergodic in 1st and 2nd moments.

3-) a)

Recall $\int_{-\infty}^{\infty} x(t)\,\delta(t-t_0)\,dt = x(t_0)$ for all $t_0 \in (-\infty,\infty)$

$$\Rightarrow \int_{-7}^{21} (t^2+1) \sum_{n=-\infty}^{\infty} \delta(t-6n)\,dt = \sum_{n} [(6n)^2+1] \quad \text{if } 6n \in (-7,21)$$

$$= \sum_{n=-1}^{3} (36n^2+1) = 545_{//}$$

b)

Recall also $\delta(at) = \dfrac{\delta(t)}{|a|}$

$$\Rightarrow \int_{-\infty}^{\infty} [e^{-4\pi t} + \cos(6\pi t)]\,\delta(2t+1)\,dt = \int_{-\infty}^{\infty} [e^{-4\pi t} + \cos 6\pi t]\,\frac{\delta(t+\frac{1}{2})}{2}\,dt$$

$$= \frac{1}{2}\left[e^{-4\pi(-\frac{1}{2})} + \cos(6\pi(-\tfrac{1}{2}))\right]$$

$$= \frac{e^{2\pi}-1}{2} \cong 267.25_{//}$$

4) Recall the following from Linear System Theory

$$A\cos(2\pi f_0 t) \longrightarrow \boxed{H(f)} \longrightarrow A|H(f_0)|\cos(2\pi f_0 t + \measuredangle H(f_0))$$

where $H(f) = |H(f)| \cdot \exp\{j \angle H(f)\}$

$$x(t) = 4\cos(30\pi t) - 3\sin(100\pi t) + 7\cos(240\pi t)$$

the output, $y(t)$, will be

$$y(t) = 4|H(15)|\cos(30\pi t + \measuredangle H(15)) - 3|H(50)|\sin(100\pi t + \measuredangle H(50))$$
$$+ 7|H(120)|\cos(240\pi t + \measuredangle H(120))$$

And

$$|H(f)| = \begin{cases} 2 & 40 \le |f| \le 100 \\ 3 & 0 \le |f| \le 40 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \measuredangle H(f) = \begin{cases} \frac{\pi}{100} f & 0 \le |f| \le 100 \\ 0 & \text{otherwise} \end{cases}$$

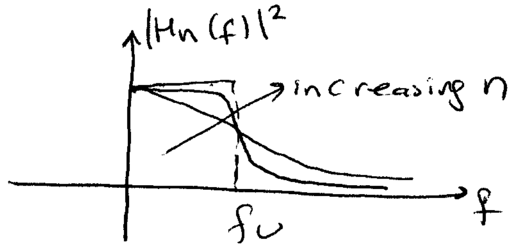
$$\Rightarrow y(t) = 4 \cdot 3 \cdot \cos\left(30\pi t + \frac{15\pi}{100}\right) - 3 \cdot 2 \cdot \sin\left(100\pi t + \frac{\pi}{2}\right)$$
$$+ 7 \cdot 0 \cdot \cos(240\pi t + 0)$$

$$= 12\cos(30\pi(t + 0.005)) - 6\cos(100\pi t)$$

p. 5

5) Part (a) of problem 1.17 in Sklar

$$|H_n(f)| = \frac{1}{\sqrt{1+\left(\frac{f}{f_u}\right)^{2n}}} \quad n \geq 1$$



where f_u is the upper -3 dB cut-off frequency.

Since $|H(f)|$ decreases monotonically with $|f|$ and $|H_n(0)| = 1$, we can write

$$10 \log_{10} |H_n(f_1)|^2 \geq -1 \quad \text{where } f_1 = 0.9 f_u$$

$$\Rightarrow \log_{10}\left(\frac{1}{1+\left(\frac{f_1}{f_u}\right)^{2n}}\right) \geq -0.1$$

$$\Rightarrow \frac{1}{1+(0.9)^{2n}} \geq 10^{-0.1} \Rightarrow 1+(0.9)^{2n} \leq 10^{0.1} = 1.2589$$

$$(0.9)^{2n} \leq 0.2589$$

$$\Rightarrow 2n \geq \frac{\log 0.2589}{\log 0.9} = 12.83$$

$$\Rightarrow \boxed{\begin{array}{c} n \geq 7 \\ n_{min} = 7 \end{array}}$$

6-) Problem 1.20 in Sklar.

$$G_x(f) = 10^{-4} \left\{ \frac{\sin \pi (f-10^6) 10^{-4}}{\pi (f-10^6) 10^{-4}} \right\}^2$$

$$= 10^{-4} \operatorname{sinc}^2 10^{-4} (f-10^6) .$$

where $\operatorname{sinc}(x) \triangleq \frac{\sin \pi x}{\pi x}$

And for BW calculation, we can ignore the Center frequency.

a) $BW = 2f_0$ where f_0 is half-power frequency

$\Rightarrow \frac{1}{2} = \operatorname{sinc}^2 f_0 10^{-4} \Rightarrow f_0 = 4.46 \text{ kHz}$

$\Rightarrow BW = 2f_0 \cong 8.92 \text{ kHz}.$

b) Noise equivalent bandwidth $= BW = \int_{-\infty}^{\infty} \operatorname{sinc}^2 (f 10^{-4}) df$

By change of integration variable, $x = f 10^{-4}$
$dx = 10^{-4} df$

$$\Rightarrow BW = \frac{1}{10^{-4}} \int_{-\infty}^{\infty} \operatorname{sinc}^2 x \, dx$$

From Parseval's Theorem $\int_{-\infty}^{\infty} \operatorname{sinc}^2 x \, dx = \int_{-\infty}^{\infty} |F\{\operatorname{sinc}(x)\}|^2 df$

and $F\{\operatorname{sinc}(x)\} = \underset{\substack{\uparrow \\ \text{unit rectangular} \\ \text{function}}}{\Pi}(f)$ $\Rightarrow \int_{-\infty}^{\infty} \operatorname{sinc}^2 x \, dx = \int_{-1/2}^{1/2} 1^2 df = 1$

$$\Rightarrow \boxed{BW = \frac{1}{10^{-4}} = 10 \text{ kHz}}$$

c) First null occurs at $f = f_0$, then

$$\text{sinc } f_0 10^{-4} = 0 \Rightarrow f_0 10^{-4} = 1 \Rightarrow f_0 = 10 \text{ kHz}$$

$$BW = 2f_0 = 20 \text{ kHz}$$

d) 99 % of power bandwidth, $BW = 2f_0$

where f_0 satisfies

$$0.99 = \frac{\cancel{(10^{-4})}\, 2 \int_0^{f_0} \text{sinc}^2 (f\, 10^{-4})\, df}{\cancel{10^{-4}} \int_{-\infty}^{\infty} \text{sinc}^2 (f\, 10^{-4})\, df}$$

by changing the integration variable such that $x = f\, 10^{-4} \Rightarrow dx = 10^{-4} df$

$$0.99 = \frac{(2)\, \cancel{10^{4}} \int_0^{f_0 10^{-4}} \text{sinc}^2 x\, dx}{\cancel{10^{4}} \underbrace{\int_{-\infty}^{\infty} \text{sinc}^2 x\, dx}_{=1}}$$

$$\Rightarrow \int_0^{f_0 10^{-4}} \text{sinc}^2 x\, dx = \frac{0.99}{2} = 0.495$$

Numerically, $f_0 10^{-4} \cong 10.2 \Rightarrow f_0 = 102 \text{ kHz}$

$$\boxed{BW = 2f_0 = 204 \text{ kHz}}$$

6-e)

$BW = 2f_o$ where f_o is the minimum frequency beyond which we have at least 35 dB attenuation.

Let $x = \pi f 10^{-4} \Rightarrow \left(\frac{\sin x}{x}\right)^2 \leq 10^{-\frac{35}{10}}$

Since $\sin^2 x$ is unity for $x = \frac{\pi}{2}(2k+1)$ $k=0,1,\dots$ the lobe beyond which the attenuation criterion is guaranteed to be met is the minimum k for which

$$10^{-3.5} \geq \frac{1}{\left(\frac{\pi}{2}(2k+1)\right)^2} \Rightarrow 35.80 \leq 2k+1 \Rightarrow \boxed{k=18}$$

Thus 18th sidelobe meets the 35 dB criterion, and actual 35 dB point will be on the falling edge of the 17th side lobe

$$\therefore \quad \pi f_o 10^{-4} > \frac{\pi}{2}(2.(17)+1) \Rightarrow f_o \geq 175\,kHz \ (1)$$

and $\quad \operatorname{sinc}^2 f_o 10^{-4} = 10^{-3.5} = 3.16\ 10^{-4} \ (2)$

$\Rightarrow$ Combine (1) & (2) and numerically we will have

$f_o \cong 175.61\ kHz \Rightarrow \boxed{BW \cong 351.2\,kHz}$

6-f) The absolute bandwidth is <u>infinite</u> since for any finite test BW, $G_x(f)$ will have positive measure beyond it.

7) $f_X(x) = 3\,e^{-6|x|}$ & $Y = 2+4X$

$$m_y = E\{y\} = E\{4+2x\} = 4 + 2m_x$$

$$m_x = \int_{-\infty}^{\infty} x\, f_x(x)\,dx = \int_{-\infty}^{\infty} x\; 3\,e^{-6|x|}\,dx = 0$$

due to oddness of integrand and limits of integration. $\Rightarrow \boxed{m_y = 4}$

$$\mathrm{Var}(y) = E\{(y-m_y)^2\} = E\{y^2\} - E^2\{y\} = \mathrm{Var}(2+4x)$$

$$= \cancel{\mathrm{Var}(2)}^{\,0} + 4^2\,\mathrm{Var}(x)$$

$$\mathrm{Var}(x) = E\{x^2\} - \cancel{E^2\{x\}}^{\,0} = \int_{-\infty}^{\infty} x^2\, 3\,e^{-6|x|}\,dx$$

$$= 6\int_0^{\infty} x^2\, e^{-6x}\,dx$$

$$\int x^2 e^{ax}\,dx = \frac{e^{ax}}{a}x^2 - \frac{2}{a^2}e^{ax}x + \frac{2}{a^3}e^{ax} \quad (1)$$

Using (1) $\displaystyle\int_0^{\infty} x^2 e^{-6x}\,dx = \cancel{-\frac{e^{-6x}}{6}x^2\Big|_0^{\infty}}^{\,0} - \cancel{\frac{2x\,e^{-6x}}{6^2}\Big|_0^{\infty}}^{\,0} - \frac{2\,e^{-6x}}{6^3}\Big|_0^{\infty}$

$$= \frac{2}{6^3} = \frac{1}{108}$$

$$\mathrm{Var}(y) = 16\,\mathrm{Var}(x) = \frac{(16).6}{108} = \frac{8}{9}\ //$$

8-)

I estimated the spectrums using the default settings of PSD command.

Figure 1 shows the spectrums in linear scale whereas Figure 2 shows them in logarithmic scale. From Figure 2, power spectrum of the input can be observed as nearly constant.

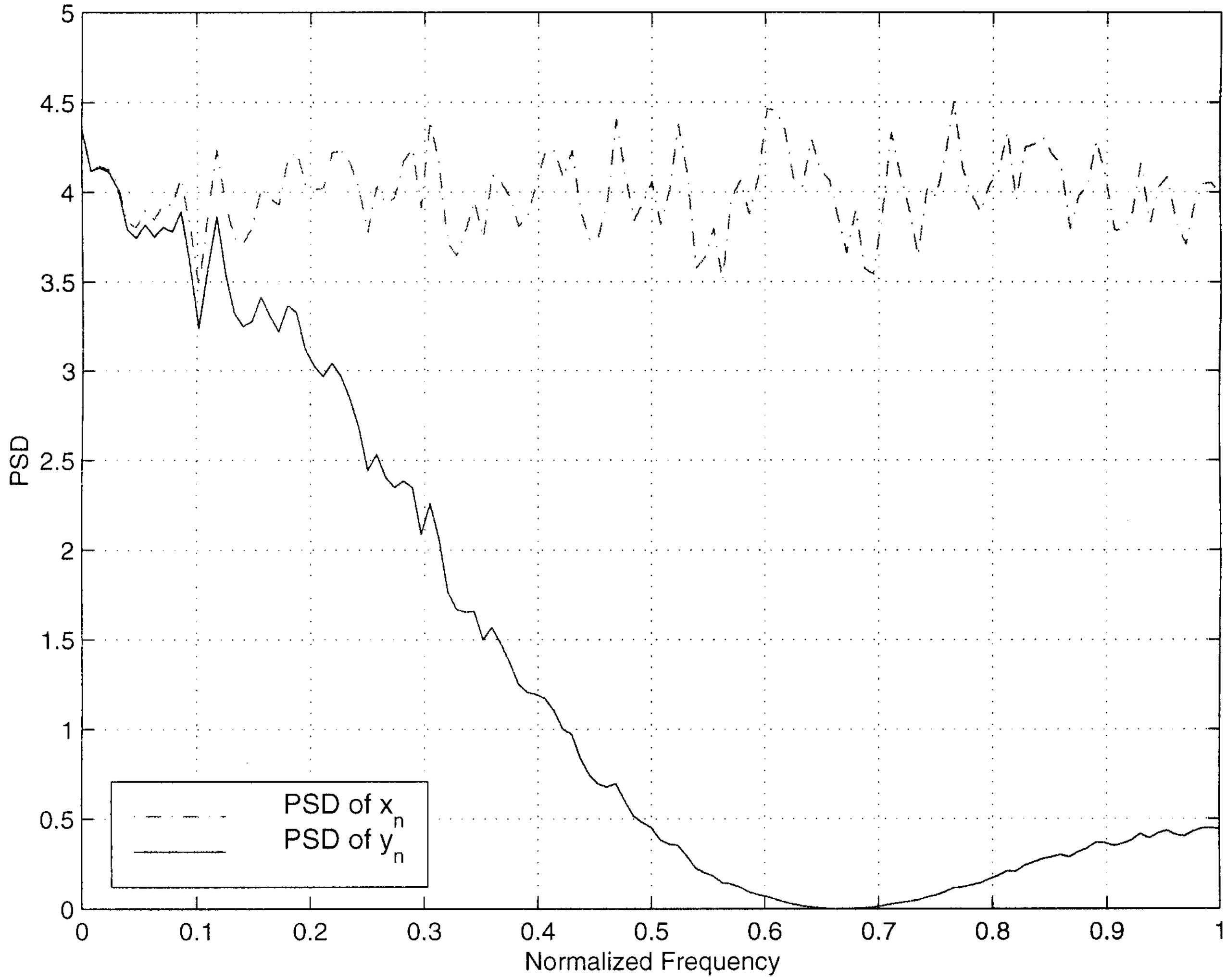
To justify these plots make sense, I tried to calculate output power spectrum, P_{yy}, by using input power spectrum, P_{xx}, and filter amplitude frequency response.

Recall $$P_{yy}(f) = P_{xx}(f)\,|H(f)|^2 \qquad (1)$$

Since $h_n = \begin{cases} 1/3 & \text{for } n=-1,0,1 \\ 0 & \text{otherwise} \end{cases} \Rightarrow H(f) = \frac{1}{3}\left(1 + 2\cos(2\pi f)\right)$

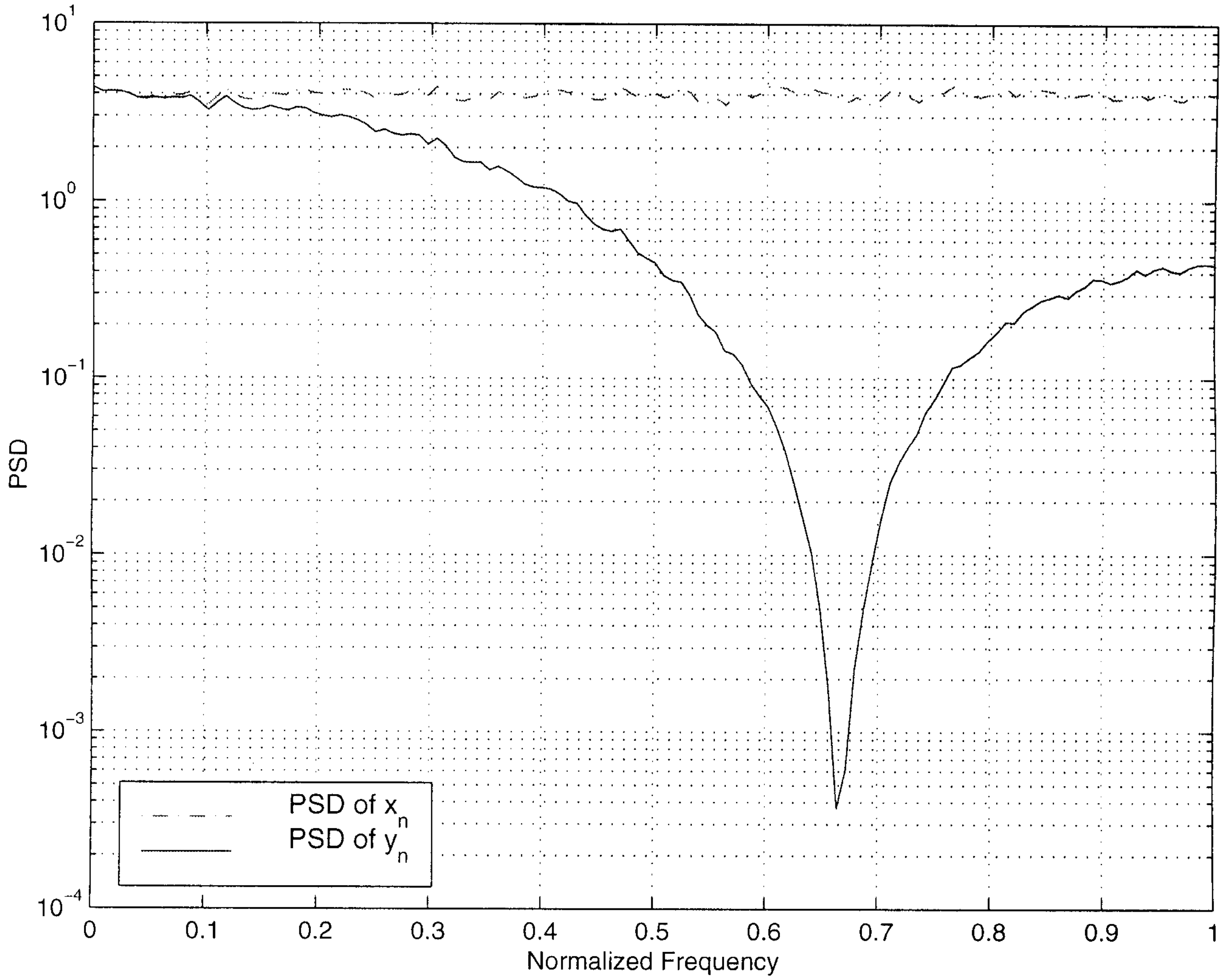
So I plotted $|H(f)|^2$ as shown in Figure 3.

And using (1), I calculated the output psd and plotted in Figure 4a. and previously estimated output psd is in Figure 4b. You can easily see that calculated and estimated output spectrum are well matched.



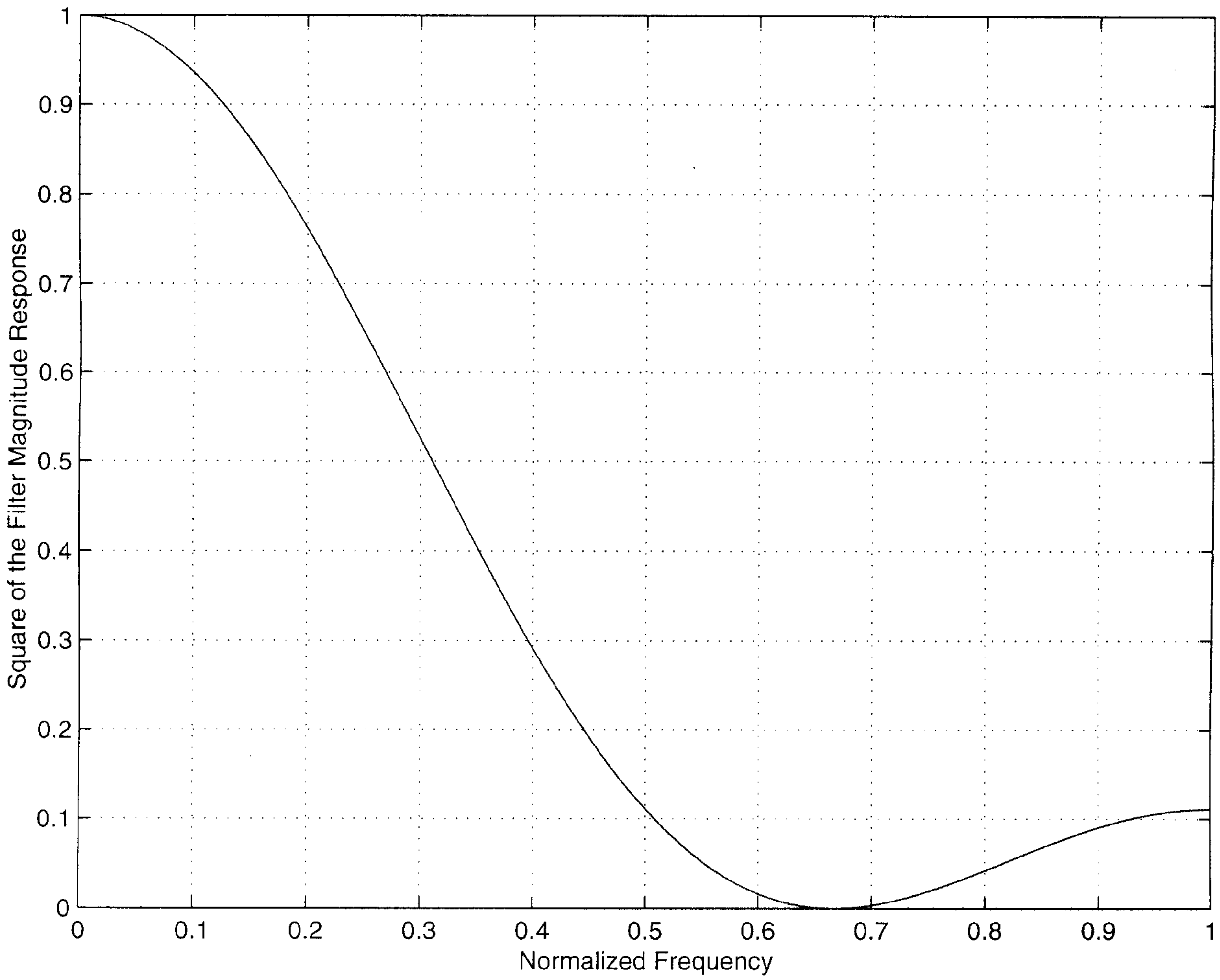
PSD of x_n
PSD of y_n
PSD
Normalized Frequency
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

Figure 1



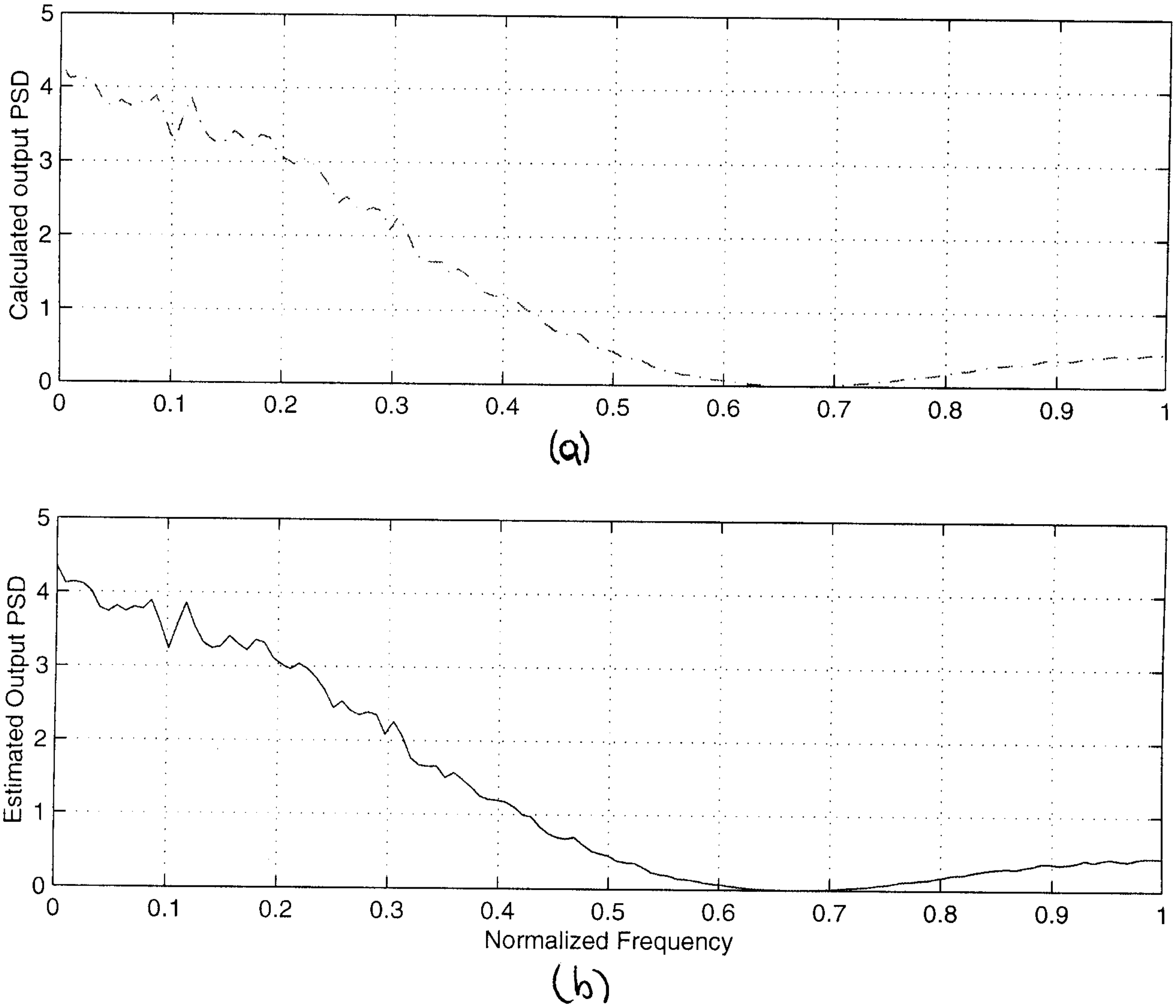
10^1
10^0
10^-1
10^-2
10^-3
10^-4
PSD
PSD of x_n
PSD of y_n
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Normalized Frequency

Figure 2



Square of the Filter Magnitude Response
Normalized Frequency
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

Figure 3



Calculated output PSD
0
1
2
3
4
5
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
(a)
Estimated Output PSD
Normalized Frequency
(b)

Figure 4

```
% Matlab Code for 18-550 HW1 Problem 8
% Written by Mehmet Keskinoz

n=1e5; %length of the xn

stdev=2; %standard deviation of xn

randn('state',sum(100*clock)); % Resetting the state of generator not to get
                               % same input sequence each time you run.

xn=stdev*randn(1,n);           % Generation of xn with zero mean and standard
                               % deviation 'stdev'

hav=[1 1 1]/3;                 % filter coefficients used to generate yn

yn=conv(xn,hav);               % generation of yn

[pxx,fx]=psd(xn);              % calculation of PSD of xn using the default
                               % setting

plot(fx,pxx,'r-.');            % plotting of PSD of xn

hold;

[pyy,fy]=psd(yn);              % calculation of PSD of yn

plot(fy,pyy);                  % plotting of PSD of yn

legend('PSD of x_n','PSD of y_n',3);    % Labeling the plots

grid;

xlabel('Normalized Frequency'); % Label x axis

ylabel('PSD');                  % Label y axis

print -dpsc hw1_q8.ps           % Print the plot

clf;                            % Clear the plot

semilogy(fx,pxx,'r-.',fy,pyy);  % Same plots in log scale

grid;

legend('PSD of x_n','PSD of y_n',3);    % Labeling the plots

xlabel('Normalized Frequency'); % Label x axis

ylabel('PSD');                  % Label y axis

print -dpsc hw1_q8log.ps        % Print the new plot
```