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PROBLEM SET 8

INTRO TO LINEAR PREDICTION

$$w[n] \rightarrow \boxed{H(z)} \rightarrow x[n]$$

$$H(z) = \frac{G}{1 - \sum_{k=1}^{p} \alpha_k z^{-k}} = \frac{G}{A(z)} = \frac{X(z)}{W(z)}$$

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

$$G w[n] = x[n] - \underbrace{\sum_{k=1}^{p} \alpha_k x[n-k]}_{\hat{x}[n]}$$

$$x[n] - \hat{x}[n] = e[n]$$

MIN $E\left[(x[n] - \hat{x}[n])^2\right]$ MMSE EST.

$$x[n] = \sum_{k=1}^{p} \alpha_k x[n-k] + G w[n]$$

MMSE EST. of $\{\alpha_k\}$ FIND $\{\alpha_k\}$ THAT MIN ε^2

Let $\varepsilon^2 = E\left[(x[n] - \hat{x}[n])^2\right]$

SOLUTION: $$\sum_{l=1}^{p} \alpha_l \phi_{xx}[l,k] = \phi_{xx}[0,k]$$

WHERE $\phi_{xx}[l,k] = E\left[x[n-l]\, x[n-k]\right]$ YULE-WALKER EQ.

SOLUTION USING AUTOCORRELATION METHOD

1. "WINDOW" DATA, CONSIDER $x[n]$, $0 \le n \le N-1$

2. CALCULATE $\hat{\phi}_{xx}[l,k]$ USING $C_{xx}[m]$

$$= \phi_{xx}[|l-k|] = \phi[|l-k|]$$

$$\sum_{l=1}^{P} \alpha_l \phi[|l-k|] = \phi[0,k]$$

FOR $P=4$

$$\begin{pmatrix} \phi[0] & \phi[1] & \phi[2] & \phi[3] \\ \phi[1] & \phi[0] & \phi[1] & \phi[2] \\ \phi[2] & \phi[1] & \phi[0] & \phi[1] \\ \phi[3] & \phi[2] & \phi[1] & \phi[0] \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} = \begin{pmatrix} \phi[1] \\ \phi[2] \\ \phi[3] \\ \phi[4] \end{pmatrix}$$

$$\underline{\underline{R}} \qquad \underline{\alpha} \qquad = \underline{P}$$

$$\underline{\underline{R}}\,\underline{\alpha} = \underline{P} \Rightarrow \boxed{\underline{\alpha} = \underline{\underline{R}}^{-1}\underline{P}}$$ WIENER-HOPF EQ

LEVINSON-DURBIN RECURSION

GIVEN $\{\phi[0], \phi[1], \ldots, \phi[P]\}$ FIND α_k, $1 \le k \le P$

IF $\underline{\underline{R}}$ IS TOEPLITZ

0. LET $E^{(0)} = \phi[0]$, LET $i=1$

1. DEFINE $k_i = \dfrac{\phi[i] - \sum_{l=1}^{i-1} \alpha_l^{(i-1)} \phi[|i-l|]}{E^{(i-1)}}$ $\quad 1 \le i \le P$

REFLECTION COEFF

$i=i+1$

2. $\alpha_i^{(i)} = k_i$

3. $\alpha_l^{(i)} = \alpha_l^{(i-1)} - k_i \alpha_{i-l}^{(i-1)}$, $\quad 1 \le l \le i-1$

4. $E^{(i)} = (1-k_i^2) E^{(i-1)}$

COMMENTS:

1. $E^{(i)}$ ENERGY IN ERROR SIGNAL

2. $\{k_i\}$ "REFLECTION COEFFS"

3. IF $H(z)$ STABLE $|k_i| < 1$

4. A.C. METHOD $\Rightarrow |k_i| < 1$

   COV METHOD $\Rightarrow |k_i|$ NOT GUARANTEED < 1

5. $\{k_i\}$ ALSO SPECIFY POLE LOCATIONS

GIVEN $\phi[0] = 5.888$

$\phi[1] = -3.52$

$\phi[2] = -1.565$

$\phi[3] = 5.167$

1. DEFINE $k_i = \dfrac{\phi[i] - \sum_{\ell=1}^{i-1} \alpha_\ell^{(i-1)} \phi[|i-\ell|]}{E^{(i-1)}}$, $1 \le i \le P$

REFLECTION COEFF

2. $\alpha_i^{(i)} = k_i$

3. $\alpha_\ell^{(i)} = \alpha_\ell^{(i-1)} - k_i \alpha_{i-\ell}^{(i-1)}$, $1 \le \ell \le i-1$

4. $E^{(i)} = (1-k_i^2) E^{(i-1)}$

$i=0$ $E^{(0)} = \phi[0] = 5.888$

$i=1$

$$k_1 = \frac{\phi[1] - \Sigma 0}{\phi[0]} = \frac{-3.52}{5.888} = -.598$$

$\alpha_1^{(1)} = -.588$

$E^{(1)} = (1-k_1^2)\phi[0] = (1-(.598)^2) 5.888 = 3.782$

$i=2$

$$k_2 = \frac{\phi[2] - \alpha_1^{(1)} \phi[1]}{E^{(1)}} = \frac{-1.565 - (-.598)(-3.52)}{3.782} = -.97$$

$\alpha_2^{(2)} = -.97$

$\alpha_1^{(2)} = \alpha_1^{(1)} - k_2 \alpha_1^{(1)} = (-.598)(1+.97) = -1.178$

$E^{(2)} = (1-k_2^2) E^{(1)} = (1-(.97)^2) 3.782 = .224$

SOME ADDITIONAL TOPICS

$$H(z) = \frac{G}{1 - \sum_{k=1}^{p} \alpha_k^{(p)} z^{-k}} = \frac{G}{A(z)}$$

Computation of Gain G

1^{st} Y-W Eq. $$\phi[0,0] = \sum_{k=1}^{p} \alpha_k \phi[0,k] + \sigma_k^2$$

$$= \sum_{k=1}^{p} \alpha_k \phi[0,k] + G^2$$

$$\boxed{G^2 = \phi[0,0] - \sum_{k=1}^{p} \alpha_k \phi[0,k]}$$

ALT. APPROACH $$w[n] \rightarrow \boxed{H(z)} \rightarrow x[n]$$

$$\frac{X(z)}{W(z)} = \frac{G}{A(z)}$$

Let $$e[n] = x[n] - \hat{x}[n] = x[n] - \sum_{k=1}^{p} \alpha_k^{(p)} x[n-k]$$

$$E(z) = \underbrace{X(z)\left(1 - \sum_{k=1}^{p} \alpha_k^{(p)} z^{-k}\right)}_{A(z)} = X(z)A(z)$$

$$H(z) = \frac{G}{A(z)} \qquad E(z) = X(z)A(z) \qquad X(z) = H(z)W(z)$$

$$A(z) = \frac{G}{H(z)} = \frac{E(z)}{X(z)}\ ; \quad G = H(z)\frac{E(z)}{X(z)} = \frac{\cancel{H(z)}\,E(z)}{\cancel{H(z)}\,W(z)}$$

$$w[n] \longrightarrow \boxed{H(z)} \longrightarrow x[n] \quad ; \quad G = \frac{E(z)}{W(z)}$$

$$e[n] = x[n] - \sum_{k=1}^{p} \alpha_k x[n-k]$$

$$G^2 = \frac{\frac{1}{N}\sum_{n=0}^{N-1} e^2[n]}{\frac{1}{N}\sum_{n=0}^{N-1} w^2[n]}$$

GIVEN $\{\alpha_k\}$ FIND $\{k_i\}$

HAVE $\alpha_k^{(i)}$ $\quad 1 \le i \le p$, $\quad 1 \le k \le i$

2. $\alpha_i^{(i)} = k_i$
3. $\alpha_l^{(i)} = \alpha_l^{(i-1)} - k_i \alpha_{i-l}^{(i-1)}$, $\quad 1 \le l \le i-1$

Let $k_p = \alpha_p^{(p)}$

For $i = p$

Let $k_i = \alpha_i^{(i)}$

$$\alpha_l^{(i-1)} = \frac{\alpha_l^{(i)} + k_i \alpha_{i-l}^{(i)}}{1-k_i^2}, \quad 1 \le l \le i-1$$

REPEAT FOR $i = p, p-1, p-2, \ldots, 1$