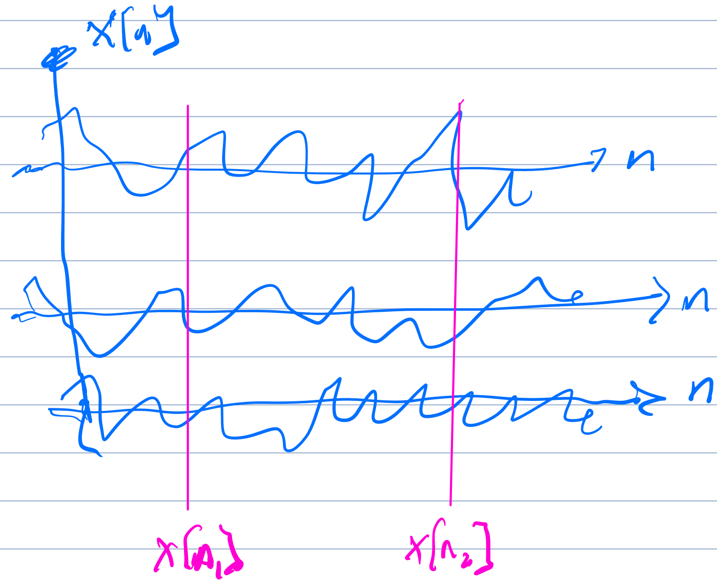
9/27/24

Recitation 5

Random Processes + Phase Vocoding



Mean $E[x[n]]$

Autocorrelation function $\phi_{xx}[n_1, n_2] = E\left[x[n_1]\, x^*[n_2]\right]$

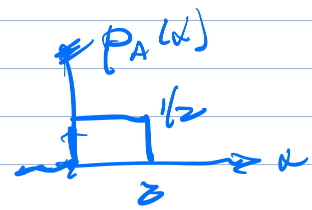
$x[n]$ WSS if

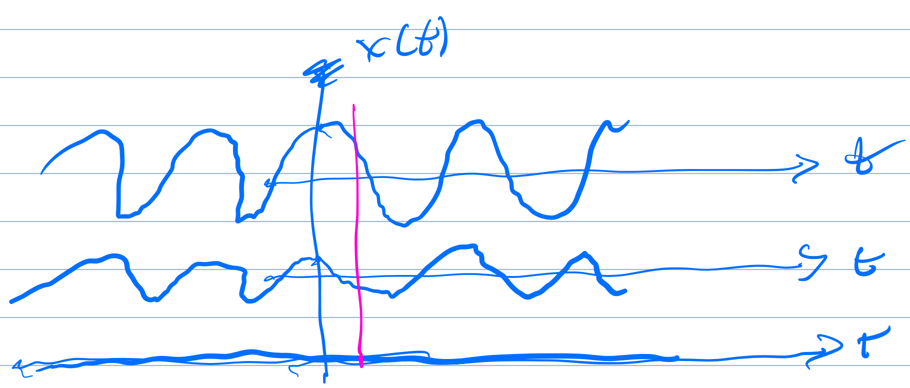
$E[x[n]]$ is constant

$\phi_{xx}[n_1, n_2]$ depends only on $n_2 - n_1 = m$, $\phi_{xx}[m]$

Random amplitude cosine

$x(t) = A \cos(\omega_0 t)$



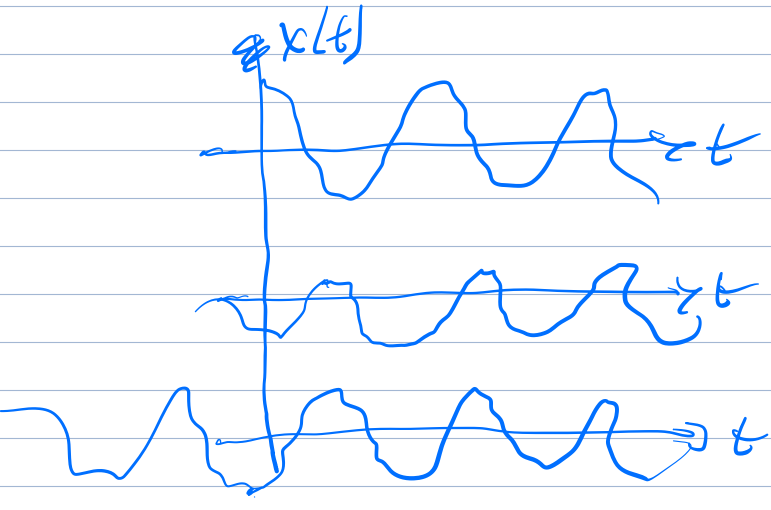


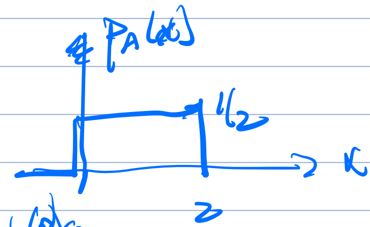
$E[x(t)] \neq$ const.

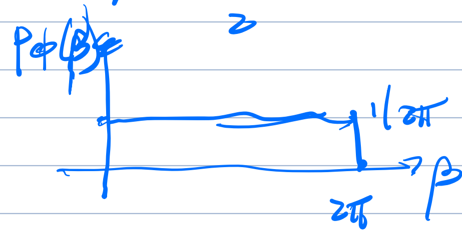
Random Amplitude + Phase

A, ϕ stat indep RVs

$$x(t) = A\cos(\omega_0 t + \phi)$$

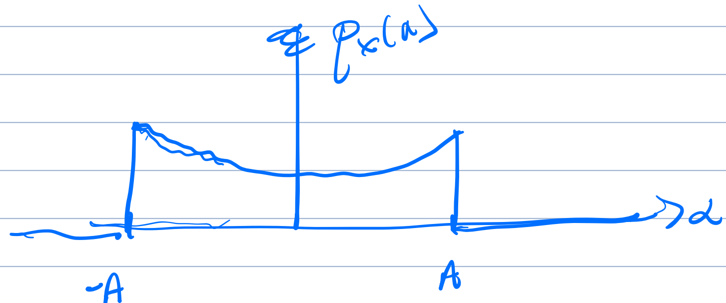






$$E[x(t)] = \overline{x(t)} = \bar{A} \cdot \overline{\cos(\omega t + \phi)}$$

Now assume A = deterministic const.



$$E[x(t)] = \int_{-A}^{A} \alpha\, p_{x(t)}(\alpha)\, d\alpha = 0$$

$$E[x(t)] = \int_0^{2\pi} E[x(t) \mid \phi = \alpha]\, p_x(\alpha)\, d\alpha$$

$$= \int_0^{2\pi} A\cos(\omega t + \alpha)\, \frac{1}{2\pi}\, d\alpha = \frac{A}{2\pi} \underbrace{\int_0^{2\pi} \cos(\omega t + \alpha)\, d\alpha}_{=0}$$

$$\phi_{xx}(t_1, t_2) = E[x(t_1)\, x(t_2)] = E[A\cos(\omega t_1 + \phi)\, A\cos(\omega t_2 + \phi)]$$

$$= \int_0^{2\pi} A\cos(\omega t_1 + \alpha)\, A\cos(\omega t_2 + \alpha)\, \underbrace{\cancel{p_\phi(\alpha)}}_{1/2\pi}\, d\alpha$$

$$\cos(x)\cos(y) = \frac{1}{2}\cos(x+y) + \frac{1}{2}\cos(x-y)$$

$$= \int_0^{2\pi} \frac{A^2}{2\pi} \frac{1}{2} \cancel{\cos(\omega(t_1+t_2)+2\alpha)}\, d\alpha + \frac{A^2}{2\pi} \int_0^{2\pi} \frac{1}{2} \cos(\omega(t_1-t_2))\, d\alpha$$

$$= 0 + \frac{A^2}{\cancel{2\pi}} \frac{1}{2} \cos(\omega(t_1-t_2)) \underbrace{\int_0^{2\pi} d\alpha}_{2\pi}$$

$$\phi_{xx}(t_1,t_2) = \frac{A^2}{2} \cos(\omega(t_1-t_2)) = \frac{A^2}{2} \cos(\omega(t_2-t_1))$$

$$= \frac{A^2}{2} \cos(\omega\tau), \quad \tau = t_2 - t_1$$

IS $x(t)$ ERGODIC ?

TIME AVERAGES

CT: $$\langle g(x(t)) \rangle = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} g(x(t))\, dt$$

DT: $$\langle g(x[n]) \rangle = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} g(x[n])$$

EX: $x(t) = A\cos(\omega t + \phi)$ A NOT RANDOM

$$\langle x(t) \rangle = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} A\cos(\omega t+\phi)\, dt = 0$$

$$\langle x(t_1) x(t_2) \rangle = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} A\cos(\omega t_1+\phi)\, A\cos(\omega t_2+\phi)\, dt$$

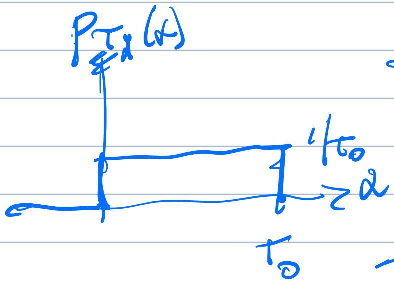
$$= \lim_{T\to\infty} \frac{A^2}{2T} \int_{-T}^{T} \left(\underbrace{\frac{1}{2}\left(\cancel{\cos(\omega(t_1+t_2)+2\phi)}\right)}_{=0} + \frac{1}{2}\cos(\omega(t_1-t_2)) \right) dt$$

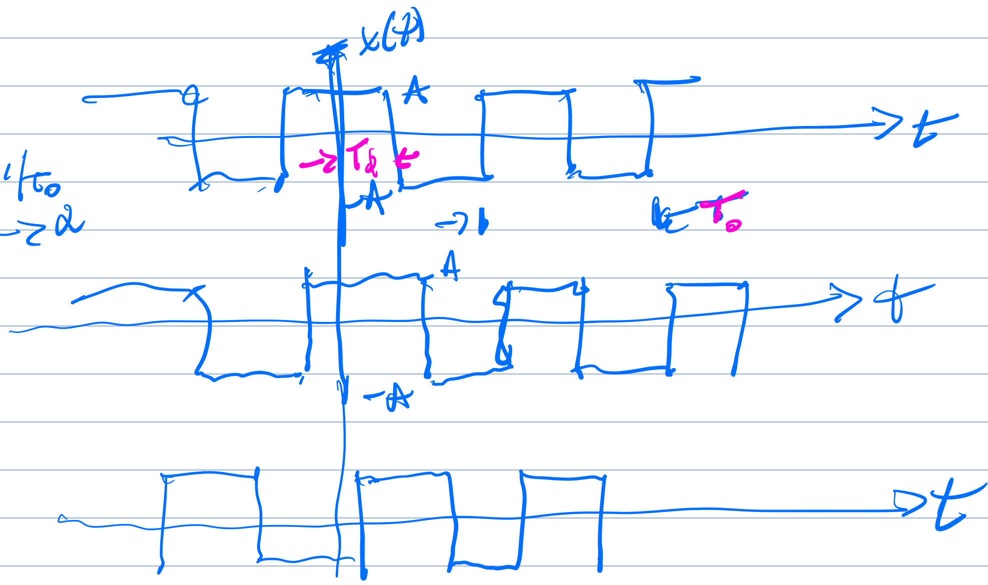
$$\langle x(t_1) x(t_2) \rangle = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} A\cos(\omega t_1+\phi)\, A\cos(\omega(t_1+t_2-t_1)+\phi)\, dt_1$$

$$= \frac{A^2}{2} \cos(t_2 - t_1)$$

[PROBLEM 5.3]

RANDOM PHASE SQ. WAVE

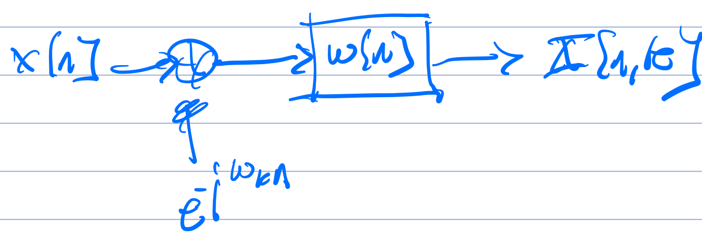


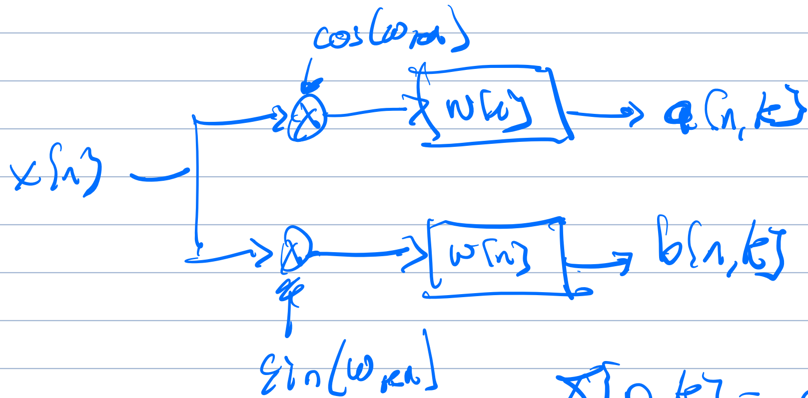


$$E[x(t_1)x(t_2)] = E[x(t_1)x(t_2) \mid x(t_1) = x(t_2)] \cdot P[x(t_1) = x(t_2)]$$
$$+ E[x(t_1)x(t_2) \mid x(t_1) \neq x(t_2)] \cdot P[x(t_1) \neq x(t_2)]$$

PHASE VOCODING

Let

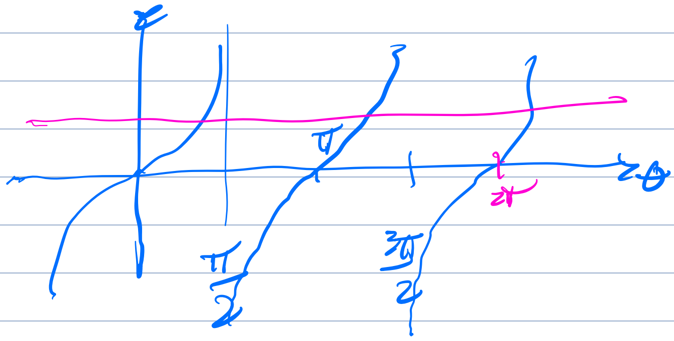




$$X[n,k] = a[n,k] - j\,b[n,k]$$

$$|X[n,k]| = \sqrt{a^2[n,k] + b^2[n,k]}$$

$$\angle X[n,k] = \theta[n,k] = -\tan^{-1}\left[\frac{b[n,k]}{a[n,k]}\right]$$



ATAN2(· , ·)

$$x_k[n] = 2\left|X[n,k]\right| \cos\left(\omega_k n + \theta[n,k]\right)$$

$$\dot\theta[n,k] \approx \frac{d}{dt}\theta(t,k)$$

$$\dot\theta[n,k] = \frac{b[n,k]\,\dot a[n,k] - a[n,k]\,\dot b[n,k]}{a^2[n,k]+b^2[n,k]}$$

COMMON APPROX: $\dot x[n] = \frac{x[n]-x[n-1]}{T}$

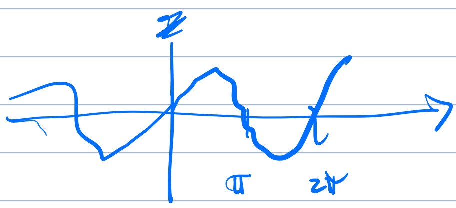
WHERE $x[n] = x(nT)$

IN CT $\dot x(t) \Leftrightarrow j\Omega$

$$\dot x[n] = \frac{x[n]-x[n-1]}{T} \Leftrightarrow \frac{1}{T}\left[X(e^{j\omega}) - e^{-j\omega}X(e^{j\omega})\right]$$

$$= \frac{1}{T}X(e^{j\omega})\left(1-e^{-j\omega}\right)$$

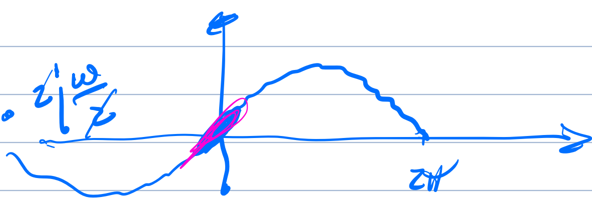
$$= \frac{1}{T}X(e^{j\omega}) \cdot e^{-j\frac{\omega}{2}}\underbrace{\left(e^{j\frac{\omega}{2}}-e^{-j\frac{\omega}{2}}\right)}_{2j\sin\left(\frac{\omega}{2}\right)}$$



$$= \frac{1}{T}X(e^{j\omega})\,e^{-j\frac{\omega}{2}}\,2j\sin\left(\frac{\omega}{2}\right)$$

FOR ω SMALL

$$\dot x[n] \Leftrightarrow \approx \frac{1}{T}X(e^{j\omega})\,e^{-j\frac{\omega}{2}} \cdot 2j\frac{\omega}{2}$$



El DIFERENCIADOR