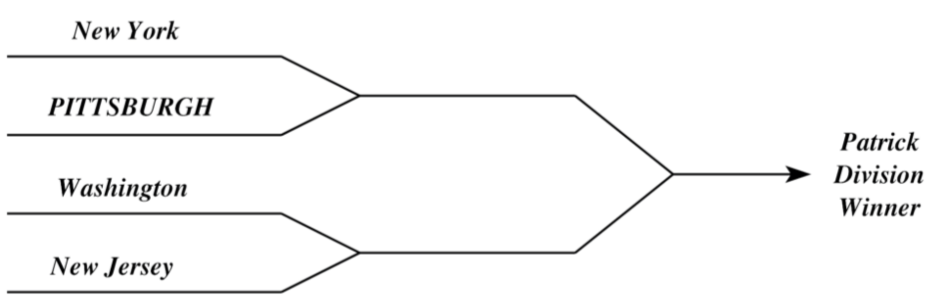
**Advanced Digital Signal Processing (18-792)**
**Fall Semester 2024**
**Department of Electrical and Computer Engineering**

# Solutions to Probability Problems in Recitation 1

**Reading:** This problem set is primarily a review of work in digital signal processing and probability theory that you should be fairly well on top of from your previous courses. If you are having a great deal of trouble with these problems you should check in with the teaching staff for remedial work.

Some of the probability theory problems in this problem set have been shamelessly plagiarized from *Fundamentals of Applied Probability Theory* by A. Drake (McGraw-Hill, 2002) which happens to have been my own undergraduate probability text. It's quite readable and in the Sorrels Engineering and Science Library, in case youd like to review probability theory from a different point of view. And I now see that free .pdf files are online on the internet .... just search. Of course, you are perfectly welcome to review whatever text you used in your studies of probability theory.

**Problem 5.1:**




As noted in the problem text, we define the probability that Team $i$ beats Team $j$ as

$$P_{ij} = \frac{\text{Number of points amassed by Team } i}{\text{Number of points amassed by Team } i + \text{Number of points amassed by Team } j}$$

(a)

$$P[4 \text{ wins}] = P[4 \text{ beats } 1]P[4 \text{ beats winner of game between 2 and 3}]$$

or

$$P[4 \text{ wins}] = P_{41}(P_{23}P_{42} + P_{32}P_{43})] = \frac{84}{187}\left(\frac{95}{180}\frac{84}{179} + \frac{85}{180}\frac{84}{169}\right) = 0.3787$$

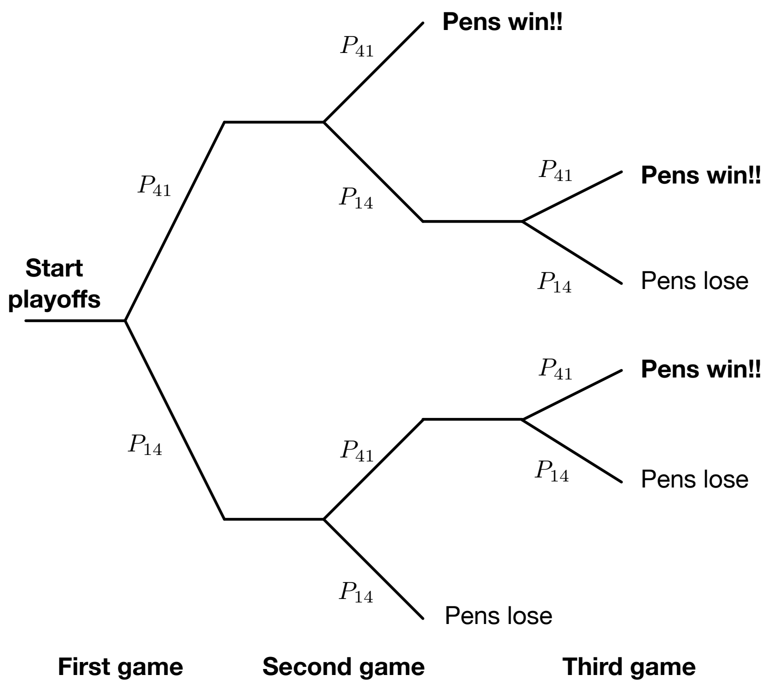
(b) Following Bayes rule,

$$P[\text{ 4 plays 3 }|\text{4 wins division }] = P[\text{4 wins division } | \text{ 4 plays 3 }]\frac{P[\text{4 plays 3 }]}{P[\text{4 wins division}]}$$

or

$$P[\text{ 4 plays 3 }|\text{4 wins division }] = \frac{(P_{43})(P_{41}P_{32})}{0.3787} = 0.6198$$

(c) The event-space diagram above depicts the various alternative events associated with the best-of-




three series, along with their associated transitional probabilities. It can be seen that the probability that the Penguins win the best-of-three series is

$$P[\text{ Pens win best of three }] = P_{41}^2(1 + 2P_{14}) = 0.4205$$

(d) The Pens are more likely to beat the Rangers in a 1-game series than in a 3-game series (which is confirmed by our numbers) because as all good sports fans know, the more games that are played, the more likely it is that the outcomes will revert to their statistically-expected values, which in this case means that the Pens will lose.

**Problem 5.4:**

We will use the notation for the probability density function (pdf)

$$p_x(\alpha) = \begin{cases} K, & a < \alpha < b \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

First, we note that the integral of $p_x(\alpha)$ must equal 1 for the function to be a legitimate pdf. Hence, by inspection $K = 1/(b-a)$.

(a) The standard deviation $\sigma_x$ of the random variable (RV) $x$ is the square root of its variance $\sigma_x^2$.

$$\sigma_x^2 = E[x^2] - (E[x])^2$$

where $E[\cdot]$ refers to the traditional statistical expectation operator. Specifically,

$$m_x \equiv E[x] = \int_{-\infty}^{\infty} \alpha p_x(\alpha) d\alpha = \frac{1}{b-a}\int_a^b \alpha d\alpha = \frac{1}{b-a}\left[\alpha^2/2\right]_a^b = \frac{1}{b-a}\frac{b^2-a^2}{2} = \frac{a+b}{2}$$

and

$$E[x^2] = \int_{-\infty}^{\infty} \alpha^2 p_x(\alpha) d\alpha = \frac{1}{b-a}\int_a^b \alpha^2 d\alpha = \frac{1}{b-a}\left[\alpha^3/3\right]_a^b = \frac{1}{b-a}\frac{b^3-a^3}{3}$$

So

$$\sigma_x = \sqrt{E[x^2] - (E[x])^2} = \sqrt{\frac{1}{3}\frac{b^3-a^3}{b-a} - \left(\frac{a+b}{2}\right)^2}$$

(b) The conditional standard deviation is computed exactly the same way as in (a), except that the integrals are only evaluated over the range of $a < \alpha \leq m_x - \sigma_x$ and $m_x + \sigma_x < \alpha \leq << b$. This makes the algebra and calculus a little messier but the principles are the same. I will not repeat these calculations here.

(c) Remember the following about multiplying by and adding constants to random variables (RVs):

- Adding a constant to an RV adds that same constant to its mean
- Multiplying a RV by a constant multiplies the mean by the same constant
- Adding a constant to an RV leaves the standard deviation (and variance) unchanged.
- Multiplyng a RV by a constant multiplies the standard deviation by that same constant

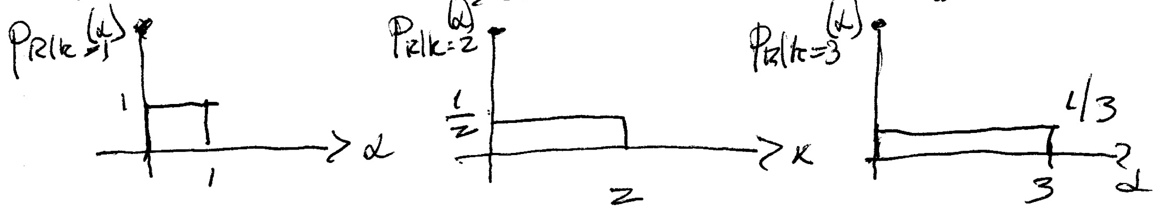
Hence, if $y = cx + d$, $E[y] = cE[x] + d$ and $\sigma_y = c\sigma_x$.

# PROBLEM 5.5

INTRO: GAMES CAN HAVE 1, 2, OR 3 SHOTS, WITH EQUAL PROBABILITY.

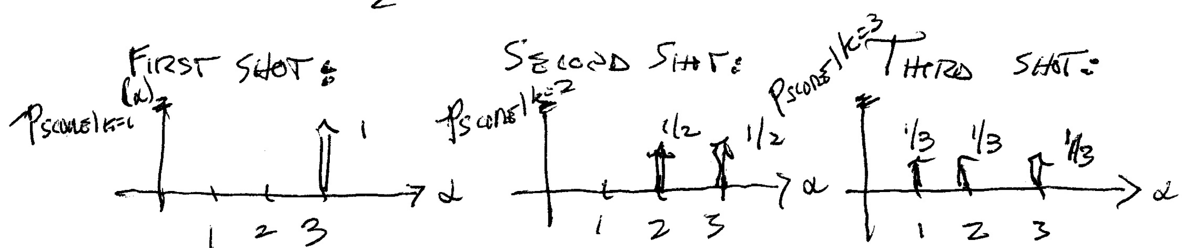
pdfs FOR RADIUS:

FIRST SHOT: SECOND SHOT: THIRD SHOT:




pdfs FOR SCORES:

FIRST SHOT: SECOND SHOT: THIRD SHOT:




NOTE THAT $E[\text{SCORE}] = 3$ FOR $k=1$, 2.5 FOR $k=2$, 2 FOR $k=3$

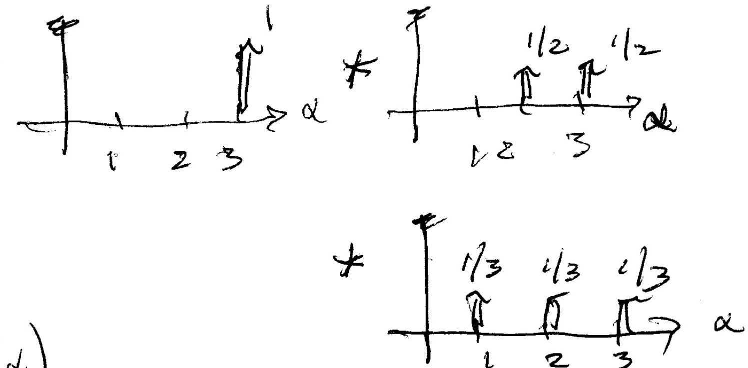
(a) $S_3$ = SCORE FROM FIRST SHOT + SCORE FROM SECOND SHOT + SCORE FROM THIRD SHOTS.

BECAUSE THE pdf FOR THE SUM OF STATISTICALLY-INDEPENDENT RANDOM VARIABLES IS THE CONVOLUTION OF THEIR INDIVIDUAL pdfs, WE OBTAIN

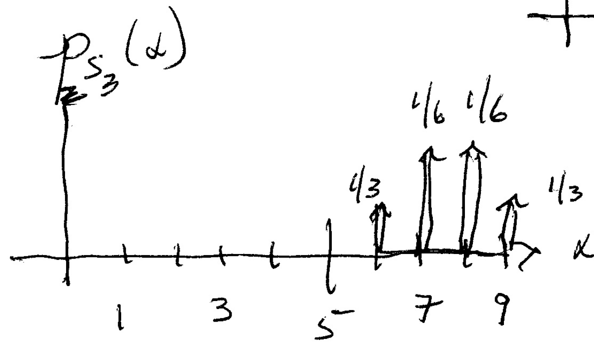
$$p_{S_3}(\alpha) = p_{score|k=1}(\alpha) * p_{score|k=2}(\alpha) * p_{score|k=3}(\alpha)$$

PROBLEM 5.5a (CONT.)

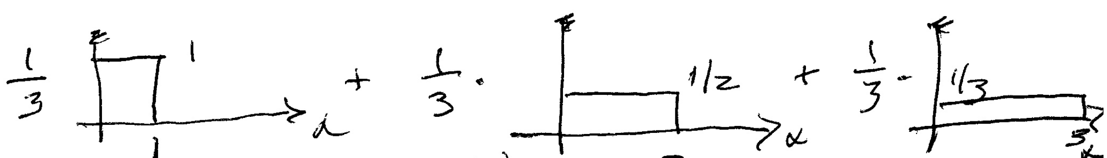
So, $p_{s_3}(\alpha) =$



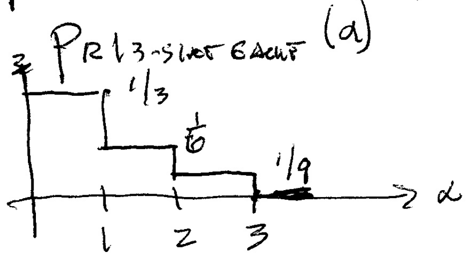

OR $p_{s_3}(\alpha)$




(b) THE PELLET COULD HAVE BEEN USED IN THE FIRST, SECOND, OR THIRD SHOT WITH EQUAL PROBABILITY. HENCE THE PDF FOR THE RADIUS IS




OR



(c) Following Bayes rule,

$$P[\text{2-shot game}|\text{score=6}] = \frac{P[\text{score=6}|\text{2-shot game}][P[\text{2-shot game}]}{P[\text{score=6}]}$$

Finding $P[\text{score} = 6]$ is a bit more complicated:

$$\begin{aligned}P[\text{score} = 6] = &P[\text{score} = 6|\text{1-shot game}]P[\text{1-shot game}] \\ &+ P[\text{score} = 6|\text{2-shot game}]P[\text{2-shot game}] \\ &+ P[\text{score} = 6|\text{3-shot game}]P[\text{3-shot game}]\end{aligned}$$

$$P[\text{1-shot game}] = P[\text{2-shot game}] = P[\text{3-shot game} = 1/3$$

Now, it is not possible to score 6 on a 1-shot game because the single first shot will have a score of 3. On a 2-shot game, a score of 6 can be achieved by scoring 3 on the first shot and 3 on the second shot, which has a probability of (1)(1/2). On a 3-shot game, a score of 6 can be achieved by scoring 3 on the first shot, 2 on the second shot, and 1 on the third shot, which has a probability of (1)(1/2)(1/3) = (1/6). Hence the total probability of scoring 6 is (0)(1/3) + (1/2)(1/3) + (1/6)(1/3) = 2/9.

As discussed above, $P[\text{score=6}|\text{2-shot game}] = 1/2$

Finally, we can compute

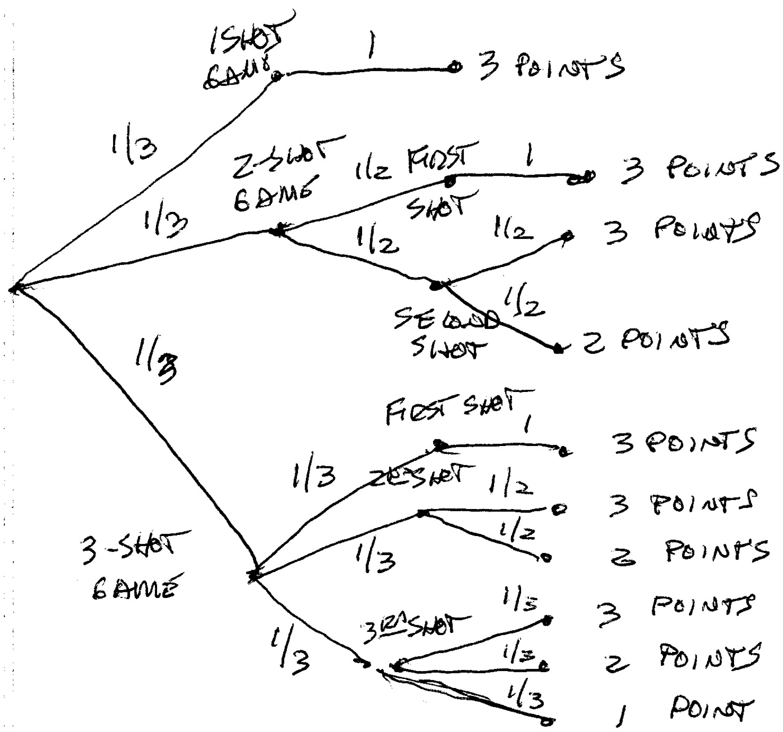
$$P[\text{2-shot game}|\text{score=6}] = \frac{P[\text{score=6}|\text{2-shot game}][P[\text{2-shot game}]}{P[\text{score=6}]} = \frac{(1/2)(1/3)}{(2/9)} = \frac{3}{4}$$

(d) IF AT LEAST ONE PELLET LANDED 15" FROM THE CENTER, WE MUST HAVE EITHER A 2-SHOT OR 3-SHOT GAME.

$$\begin{aligned}E[\text{SCORE} | \text{2-SHOT OR 3-SHOT GAME}] &= P[\text{2-SHOT GAME}]\, E[\text{SCORE} | \text{2-SHOT GAME}] \\ &+ P[\text{3-SHOT GAME}] \cdot E[\text{SCORE} | \text{3-SHOT GAME}] \\ &= \left(\tfrac{1}{3}\right)(5.5) + \left(\tfrac{1}{3}\right)(7.5) = \tfrac{13}{3} = 4.33\end{aligned}$$

PROBLEM 5.5 (CONT.)

(e) CONSIDER THE FOLLOWING EVENT SPACE:




PROBABILITY OF 3 POINTS $= \frac{1}{3} + \frac{1}{6} + \frac{1}{12} + \frac{1}{9} + \frac{1}{18} + \frac{1}{27} = .787$

PROBABILITY OF 2 POINTS $= \frac{1}{12} + \frac{1}{18} + \frac{1}{27} = .1759$

PROBABILITY OF 1 POINT $= \frac{1}{27} = .037$

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