11/6/24

INTRO TO ADAPTIVE FILTERING

(LD 5.0–5.5, NOTES 8.1–8.6)

QUIZ 2 11/20, COVERAGE THROUGH PS 9

CONVENTIONAL FILTER DESIGN

1. DEVELOP SPECS FOR FILTER (APPLICATION DRIVEN)
2. DESIGN FILTER (OS 4P CHAP 7)
3. IMPLEMENT FILTER (OS 4P CHAP 6)

<u>COEFFICIENTS FIXED</u>

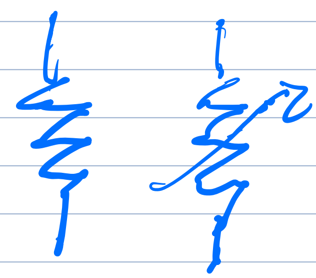
ADAPTIVE FILTERING COEFFICIENTS <u>NOT</u> FIXED

DESIGN BASED ON FUNCTION

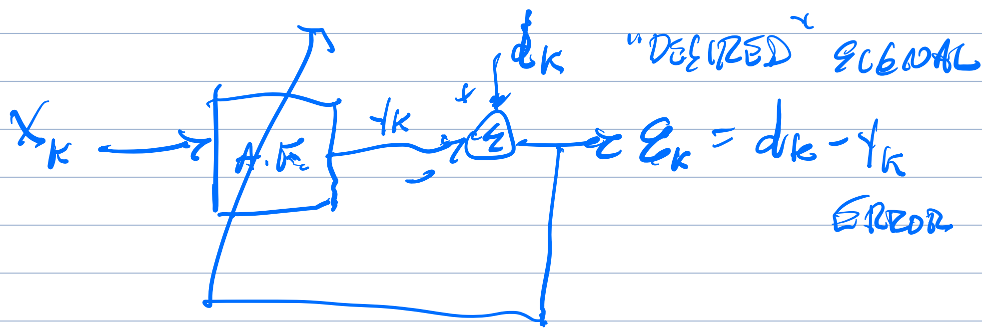
NOTATIONAL CONVENTIONS

LSI SYSTEM $x[n] \rightarrow \boxed{h[n]} \rightarrow y[n]$

ADAPTIVE FILTERS $x_k \rightarrow \boxed{h_k} \rightarrow y_k$



STANDARD A.F. STRUCTURE



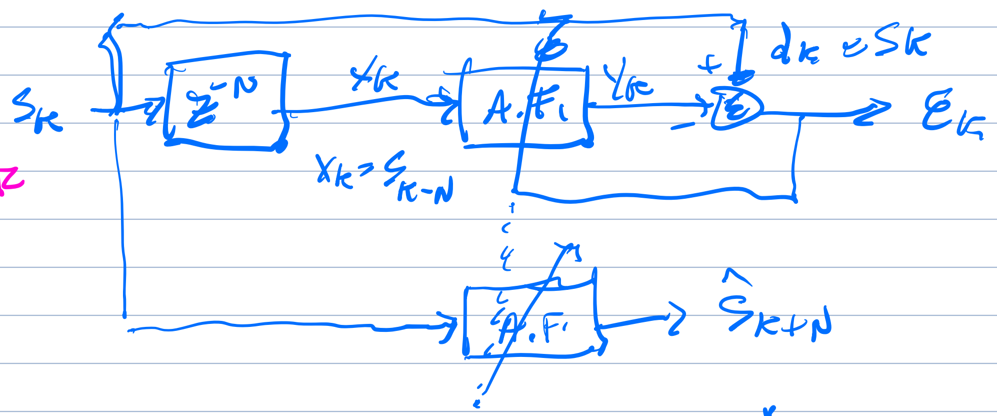
MANIPULATE FILTER COEFFS TO MINIMIZE

$$\xi^2 = E[\varepsilon_k^2] = E[(d_k - y_k)^2] \quad \text{MMSE}$$

SOME SIMPLE EXAMPLES

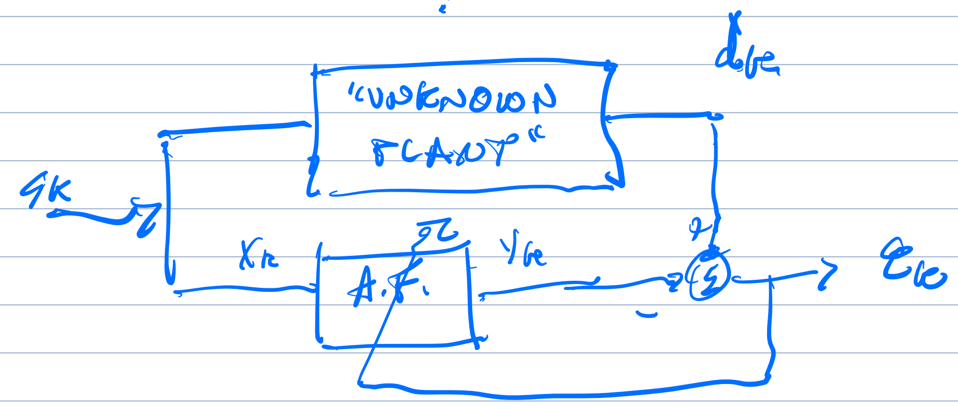
Ex. 1

ADAPTIVE PREDICTOR



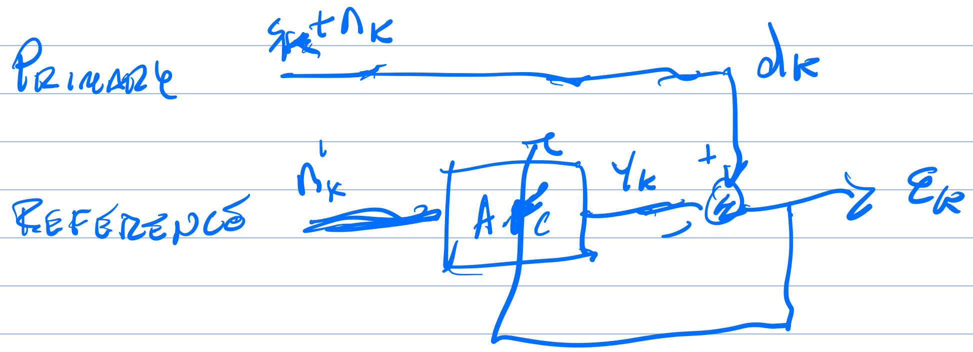
Ex 2

SYSTEM ID.



Ex 3

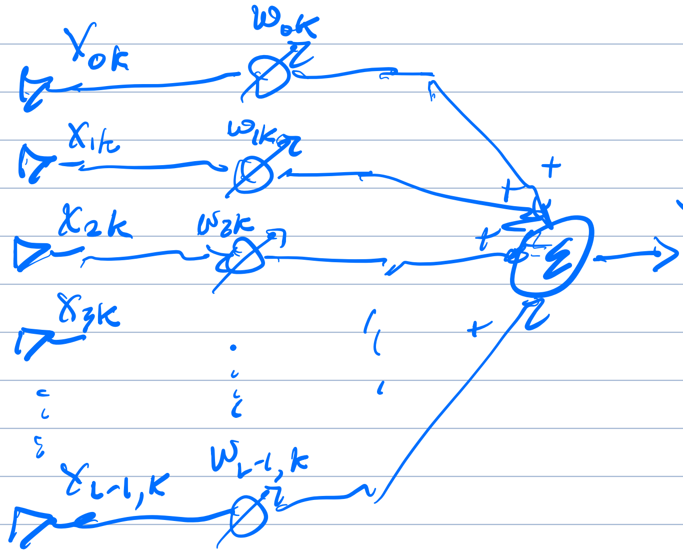
ADAPTIVE NOISE CANCELLATION



ADAPTIVE LINEAR COMBINER

EX 1
SENSOR ARRAY

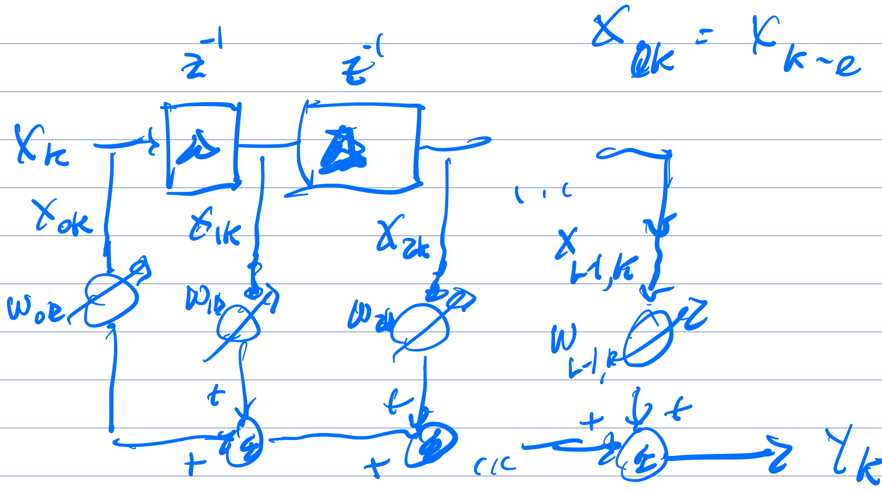
L SENSORS

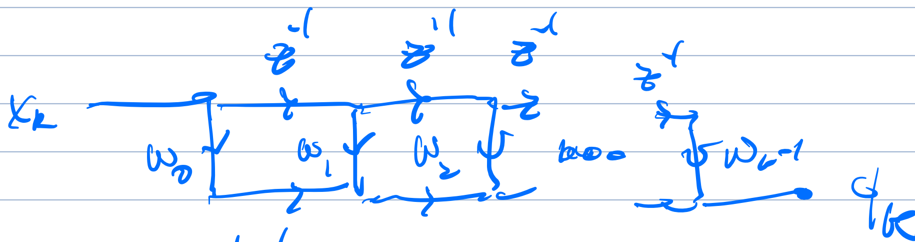


$$y_k = \sum_{\ell=0}^{L-1} x_{\ell k} w_{\ell k}$$

EX 2
FIR FILTER
TAPPED DELAY LINE
(TDL)

$$x_{\ell k} = x_{k-\ell}$$





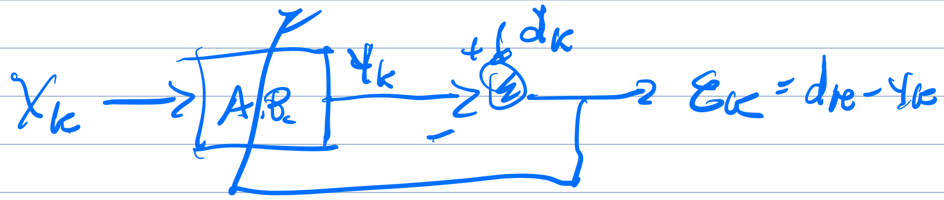
$$y_k = \sum_{\ell=0}^{L-1} x_{\ell k} w_{\ell k} = \sum_{\ell=0}^{L-1} x_{k-\ell} w_{\ell k}$$

MATRIX-VECTOR FORM

Let $\underline{W_k} \triangleq \begin{pmatrix} w_{0k} \\ w_{1k} \\ w_{2k} \\ \vdots \\ w_{L-1,k} \end{pmatrix}$ $\quad \underline{X_k} \triangleq \begin{pmatrix} x_{0k} \\ x_{1k} \\ x_{2k} \\ \vdots \\ x_{L-1,k} \end{pmatrix}$

$$y_k = \sum_{l=0}^{L-1} x_{lk} w_{lk} = \underline{w}_k^T \underline{x}_k = \underline{x}_k^T \underline{w}_k$$

THE "PERFORMANCE FUNCTION"



$$\varepsilon_k = d_k - y_k = d_k - \underline{x}_k^T \underline{w}_k = d_k - \underline{w}_k^T \underline{x}_k$$

FIND $\xi^2 = E[\varepsilon_k^2]$ AS FN. OF $\underline{w}_k$

$$\xi^2 = E[\varepsilon_k^2] = E\{(d_k - y_k)^2\} = E[d_k^2] + E[\underbrace{\underline{w}_k^T x_k}_{y_k} \underbrace{x_k^T \underline{w}_k}_{y_k}]$$

$$- 2E[d_k \underline{x}_k^T \underline{w}_k]$$

DEFINE $\underline{\underline{R}} \triangleq E[\underline{x}_k \underline{x}_k^T]$ $L \times L$

$\underline{P} \triangleq E[d_k \underline{x}_k]$ $L \times 1$

$$\xi^2 = E[d_k^2] + \underline{w}_k^T \underline{\underline{R}}\, \underline{w}_k - 2P^T \underline{w}_k$$

QUADRATIC FUNCTION of $\underline{w}_k$

TO FIND MIN.

$$\underline{\nabla}_{\xi^2} = \frac{\partial \xi^2}{\partial \underline{w}_k} = \begin{Bmatrix} \frac{\partial \xi^2}{\partial w_0} \\ \frac{\partial \xi^2}{\partial w_1} \\ \frac{\partial \xi^2}{\partial w_2} \\ \vdots \\ \frac{\partial \xi^2}{\partial w_{L-1}} \end{Bmatrix} = 0,$$

SOLVE FOR $\underline{w}_k$

$$\xi^2 = E[d_k^2] + \underline{w}_k^T \underline{\underline{R}}\, \underline{w}_k - 2\underline{P}^T \underline{w}_k$$

$$\nabla_{\xi^2} = \frac{\partial \xi^2}{\partial \underline{w}_k} = \cancel{2}\underline{\underline{R}}\,\underline{w} - \cancel{2}\underline{P} = 0$$

$$\underline{\underline{R}} = E[\underline{x}_k \underline{x}_k^T] \qquad \underline{\underline{R}}\,\underline{w} = \underline{P}$$

$$\underline{P} = E[d_k \underline{x}_k] \qquad \underline{\underline{R}}^{-1}\underline{\underline{R}}\,\underline{w} = \underline{\underline{R}}^{-1}\underline{P}$$

$$\boxed{\underline{w} = \underline{\underline{R}}^{-1}\underline{P}}$$

WIENER HOPF EQ.

WBG $\underline{\alpha} = \underline{\underline{R}}^{-1}\underline{\phi}$

NEWTON SOLUTION

OPTIMAL $\underline{w} \triangleq \underline{w}^*$

$$\nabla_{\xi^2} = 2\underline{\underline{R}}\,\underline{w} - 2\underline{P}$$

FOR $\underline{w} = \underline{\underline{R}}^{-1}\underline{P} \triangleq \underline{w}^*$

PREMULT BY $\frac{1}{2}\underline{\underline{R}}^{-1}$,

$$\frac{1}{2}\underline{\underline{R}}^{-1}\nabla_{\xi^2} = \frac{1}{\cancel{2}}\underline{\underline{R}}^{-1}\cancel{2}\cancel{\underline{\underline{R}}}\,\underline{w} - \underbrace{\frac{1}{\cancel{2}}\underline{\underline{R}}^{-1}\cancel{2}\underline{P}}_{\underline{w}^*}$$

$$\frac{1}{2}\underline{\underline{R}}^{-1}\nabla_{\xi^2} = \underline{w} - \underline{w}^*$$

$$\boxed{\underline{w}^* = \underline{w} - \frac{1}{2}\underline{\underline{R}}^{-1}\nabla_{\xi^2}}$$

IN PRACTICE, USE GRADIENT DESCENT TO SOLVE

$$\underline{W}_{k+1} = \underline{W}_k - \mu \underline{\underline{R}}^{-1} \underline{\nabla}_{\xi}$$

$$\underline{W}_{k+1} = \underline{W}_k - \mu \underline{\underline{R}}^{-1} \underbrace{\left(2\underline{\underline{R}}\,\underline{W} - 2\underline{P}\right)}_{\nabla_{\xi^2}}$$

SIMPLE SOLUTION (WIDROW)

1. IGNORE $\underline{\underline{R}}^{-1}$
2. REPLACE $\underline{\underline{R}} = E\left[\underline{X}\,\underline{X}^T\right]$ BY $\underline{X}_k \underline{X}_k^T$

   REPLACE $\underline{P} = E\left[d_k \underline{X}_k\right]$ BY $d_k \underline{X}_k$

THIS PRODUCES $\underline{W}_{k+1} = \underline{W}_k - \mu \underline{\nabla}$

$$\underline{W}_{k+1} = \underline{W}_k - \mu\left(2\underline{\underline{R}}\,\underline{W} - 2\underline{P}\right)$$

$$W_{k+1} = W_k - \mu\left(2\underline{X}_k \underline{X}_k^T \underline{W}_k - 2 d_k \underline{X}_k\right)$$

$$= \underline{W}_k - \mu \underline{X}_k \left(2 y_k - 2 d_k\right)$$

$$= \underline{W}_k + 2\mu \underline{X}_k \left(d_k - y_k\right)$$

LMS ALGORITHM

$$\boxed{W_{k+1} = W_k + 2\mu \underline{X}_k \varepsilon_k}$$