10/30/24 INTRO TO LINEAR PREDICTION

[RS 8.1–8.2, OS48 6.6, NOTES 7.1–7.5]

REMINDER - QUIZ 2 11/20, COVERAGE THROUGH PS 8 OR 9 (?)

GIVEN WSS RP $x[n]$

GIVEN $\{x[n-1], x[n-2], \ldots x[n-P]\}$

ESTIMATE $\hat{x}[n] = \sum_{\ell=1}^{P} \alpha_\ell \, x[n-\ell]$; LET $e[n] = x[n] - \hat{x}[n]$

FIND $\{\alpha_k\}$ THAT MINIMIZES $\xi^2 = E\left[e^2[n]\right]$

$$\min \xi^2 = E\Big[\underbrace{\left(x[n] - \hat{x}[n]\right)^2}_{e^2[n]}\Big] = \text{MMSE}$$

TO FIND $\{\hat{\alpha}_k\}$, COMPUTE $\dfrac{d\xi^2}{d\alpha_k} = 0$, SOLVE FOR α_k

$$\frac{d\xi^2}{d\alpha_k} = \frac{d}{d\alpha_k} = 0 = E\left[\left(x[n] - \sum_{\ell=1}^{P} \alpha_\ell \, x[n-\ell]\right)^2\right]$$

LET $\phi_{xx}[i,j] = E\left[x[n-i]\,x[n-j]\right]$

$$\frac{d\xi^2}{d\alpha_k} = E\left[\frac{d}{d\alpha_k}\left(x[n] - \sum_{\ell=1}^{P} \alpha_\ell \, x[n-\ell]\right)^2\right]$$

$$= E\left[2\left(x[n] - \sum_{\ell=1}^{P} \alpha_\ell \, x[n-\ell]\right) x[n-k]\right] = 0$$

$$\cancel{2}\left(\phi_{xx}[0,k] - \sum_{\ell=1}^{P} \alpha_\ell \, \phi_{xx}[\ell,k]\right) = 0$$

$$\boxed{\sum_{\ell=1}^{P} \alpha_\ell \, \phi_{xx}[\ell,k] = \phi_{xx}[0,k]}$$

2ND YULE-WALKER EQ.

SOLUTION FOR $P=4$ Let $\phi[i,j] = \phi_{xx}[i,j] = E\left[x[n-i]\,x[n-j]\right]$

$k=1$ $\alpha_1 \phi[1,1] + \alpha_2 \phi[2,1] + \alpha_3 \phi[3,1] + \alpha_4 \phi[4,1] = \phi[0,1]$

$k=2$ $\alpha_1 \phi[1,2] + \alpha_2 \phi[2,2] + \alpha_3 \phi[3,2] + \alpha_4 \phi[4,2] = \phi[0,2]$

$k=3$ $\alpha_1 \phi[1,3] + \alpha_2 \phi[2,3] + \alpha_3 \phi[3,3] + \alpha_4 \phi[4,3] = \phi[0,3]$

$k=4$ $\alpha_1 \phi[1,4] + \alpha_2 \phi[2,4] + \alpha_3 \phi[3,4] + \alpha_4 \phi[4,4] = \phi[0,4]$

$$\underbrace{\begin{pmatrix} \phi[1,1] & \phi[2,1] & \phi[3,1] & \phi[4,1] \\ \phi[1,2] & \phi[2,2] & \phi[3,2] & \phi[4,2] \\ \phi[1,3] & \phi[2,3] & \phi[3,3] & \phi[4,3] \\ \phi[1,4] & \phi[2,4] & \phi[3,4] & \phi[4,4] \end{pmatrix}}_{\underline{\underline{R}}} \underbrace{\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix}}_{\underline{\alpha}} = \underbrace{\begin{pmatrix} \phi[0,1] \\ \phi[0,2] \\ \phi[0,3] \\ \phi[0,4] \end{pmatrix}}_{\underline{P}}$$

$$\underline{\underline{R}}\,\underline{\alpha} = \underline{P}$$

$$\cancel{\underline{\underline{R}}^{-1}\underline{\underline{R}}}\,\underline{\alpha} = \underline{\underline{R}}^{-1}\underline{P}$$

$$\boxed{\underline{\alpha} = \underline{\underline{R}}^{-1}\underline{P}}$$

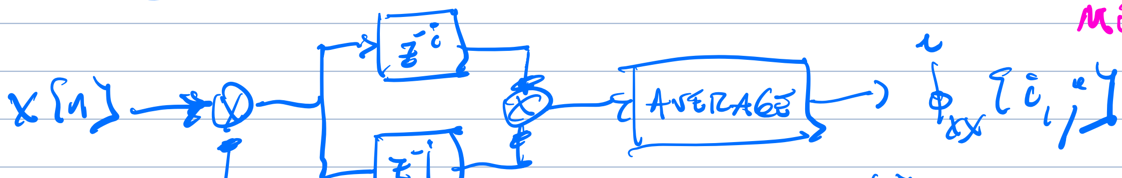
WIENER-HOPF EQ.

ESTIMATING $\phi_{xx}[i,j]$

GIVEN $x[n]$, $0 \le n \le N-1$

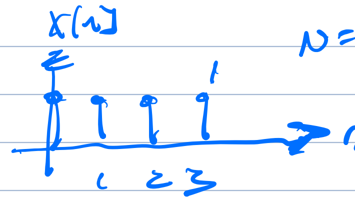
FIND $\phi_{xx}[i,j]$ FOR $|i| \le N$, $|j| \le n$

METHOD I: WINDOW THEN CORRELATE "AUTO-CORRELATION METHOD"



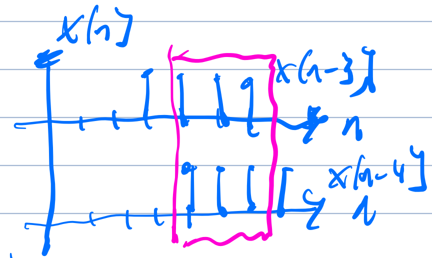
$w[n] = \begin{cases} 1, & 0 \le n \le N-1 \\ 0, & \text{ELSE} \end{cases}$

CONSIDER



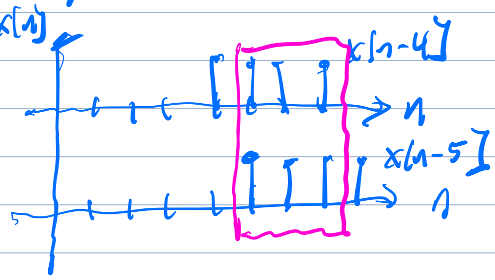
$N=4$

$\phi_{xx}[3,4]$



$\phi_{xx}[i,j] = \phi_{xx}[|j-i|]$

$\phi_{xx}[4,5]$



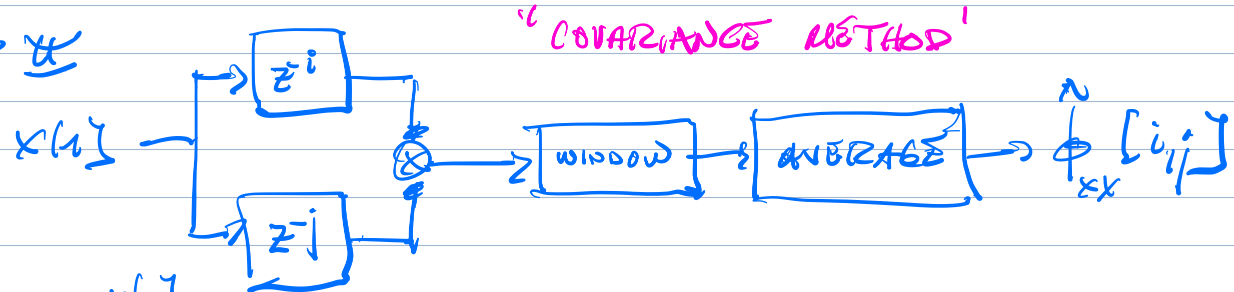
$$\begin{pmatrix} \phi[1,1] & \phi[2,1] & \phi[3,1] & \phi[4,1] \\ \phi[1,2] & \phi[2,2] & \phi[3,2] & \phi[4,2] \\ \phi[1,3] & \phi[2,3] & \phi[3,3] & \phi[4,3] \\ \phi[1,4] & \phi[2,4] & \phi[3,4] & \phi[4,4] \end{pmatrix} = \begin{pmatrix} \phi[0] & \phi[1] & \phi[2] & \phi[3] \\ \phi[1] & \phi[0] & \phi[1] & \phi[2] \\ \phi[2] & \phi[1] & \phi[0] & \phi[1] \\ \phi[3] & \phi[2] & \phi[1] & \phi[0] \end{pmatrix}$$

CAN BE SOLVED USING LEVINSON-DURBIN RECURSION

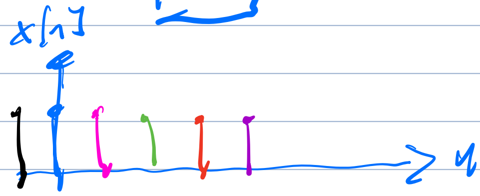
TOEPLITZ

METHOD II

"COVARIANCE METHOD"

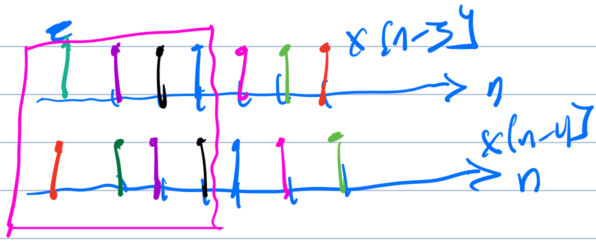


Let N=4

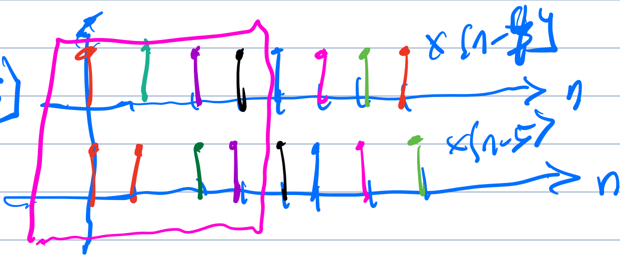


$\phi_{xx}[3,4] \neq \phi_{xx}[4,5]$

$\phi_{xx}[3,4]$



$\phi_{xx}[4,5]$



$$\underline{\underline{R}} = \begin{pmatrix} \phi[1,1] & \phi[2,1] & \phi[3,1] & \phi[4,1] \\ \phi[1,2] & \phi[2,2] & \phi[3,2] & \phi[4,2] \\ \phi[1,3] & \phi[2,3] & \phi[3,3] & \phi[4,3] \\ \phi[1,4] & \phi[2,4] & \phi[3,4] & \phi[4,4] \end{pmatrix}$$

$\phi[2,1] = \phi[1,2]$

HERMITIAN SYMMETRIC

COVARIANCE (OR HERMITIAN) SYMMETRY

BUT $\underline{\underline{\text{NOT}}}$ TOEPLITZ

NOT TOEPLITZ

INVERT $\underline{\underline{R}}$ USING CHOLESKY DECOMPOSITION

SOLVING WIENER-HOPF EQ. USING LEVINSON-DURBIN (LD) RECURSION

$$\begin{pmatrix} \phi[1,1] & \phi[2,1] & \phi[3,1] & \phi[4,1] \\ \phi[1,2] & \phi[2,2] & \phi[3,2] & \phi[4,2] \\ \phi[1,3] & \phi[2,3] & \phi[3,3] & \phi[4,3] \\ \phi[1,4] & \phi[2,4] & \phi[3,4] & \phi[4,4] \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} = \begin{pmatrix} \phi[0,1] \\ \phi[0,2] \\ \phi[0,3] \\ \phi[0,4] \end{pmatrix}$$

$$\underline{\underline{R}} \qquad \underline{\alpha} = \underline{P}$$

GIVEN $\{\phi_{xx}[m]\}$, $|m| \le P$

FIND $\{\alpha_k\}$, $k = 1 - P$

$$H(z) = \frac{G}{1 - \sum_{k=1}^{P} \alpha_k^{(P)} z^{-k}}$$

LD RECURSION

0. LET $E^{(0)} = \phi[0]$

1. $k_i = \dfrac{\phi[i] - \sum_{\ell=1}^{i-1} \alpha_\ell^{(i-1)} \phi[i-\ell]}{E^{(i-1)}}$

"REFLECTION COEFFICIENTS"

2. $\alpha_i^{(i)} = k_i$

INCREASE i

3. $\alpha_\ell^{(i)} = \alpha_\ell^{(i-1)} - k_i \alpha_{i-\ell}^{(i-1)}$, $1 \le \ell \le i-1$

4. $E^{(i)} = (1-k_i^2) E^{(i-1)}$

IF $H(z)$ STABLE, $|k_i| < 1$

0. LET $E^{(0)} = \phi[0]$

1. $k_i = \dfrac{\phi[i] - \sum_{\ell=1}^{i-1} \alpha_\ell^{(i-1)} \phi[i-\ell]}{E^{(i-1)}}$

"REFLECTION COEFFICIENTS"

2. $\alpha_i^{(i)} = k_i$

3. $\alpha_\ell^{(i)} = \alpha_\ell^{(i-1)} - k_i \alpha_{i-\ell}^{(i-1)}$, $1 \le \ell \le i-1$

4. $E^{(i)} = (1-k_i^2) E^{(i-1)}$

Ex... GIVEN $\{\phi[m]\}$

FIND α_k

$i=1$

$E^{(0)} = \phi[0]$

1. $K_1 = \dfrac{\phi[1] - 0}{\phi[0]} = \dfrac{\phi[1]}{\phi[0]}$

2. $\alpha_1^{(1)} = K_1 = \dfrac{\phi[1]}{\phi[0]}$

4. $E^{(1)} = (1-k_1^2) E^{(0)} = \left(1 - \left(\dfrac{\phi[1]}{\phi[0]}\right)^2\right) \phi[0]$

$i=2$

1. $k_2 = \dfrac{\phi[2] - \alpha_1^{(1)} \phi[1]}{E^{(1)}} = \dfrac{\phi[2] - \dfrac{\phi[1]}{\phi[0]}}{1 - \left(\dfrac{\phi[1]}{\phi[0]}\right)^2 \phi[0]}$

2. $\alpha_2^{(2)} = k_2$

3. $\alpha_1^{(2)} = \alpha_1^{(1)} - k_2 \alpha_1^{(1)} = \alpha_1^{(1)} (1-k_2)$

4. $E^{(2)} = (1-k_2^2) E^{(1)}$

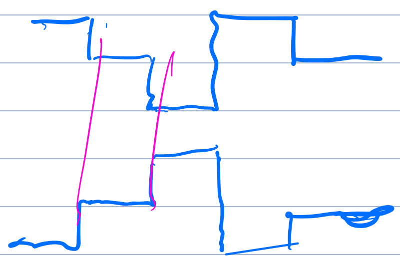
$i=3$

1. $k_3 = \dfrac{\phi[3] - \sum_{\ell=1}^{2} \alpha_\ell^{(2)} \phi[i-\ell]}{E^{(2)}}$

COMMENTS:

1. IF $|k_i| < 1 \Rightarrow$ SYSTEM STABLE

2. W/ A.C. METHOD, $|k_i| < 1$, ALWAYS

3. 

4. $\{k_i\}$ REFLECTION COEFFS

$$\alpha_\ell^{(i)} = \alpha_\ell^{(i-1)} - k_i \alpha_{i-\ell}^{(i-1)}$$

GIVEN $\{k_i\}$ CAN OBTAIN $\{\alpha_i\}$

" $\{\alpha_i\}$ " " $\{k_i\}$