10/28/24 MEM SPECTRAL ESTIMATION,

INTRO TO LINEAR PREDICTION

(ROBINSON PAPER, NOTES 6.3–7.2, RABINER + SCHAFER 8.1–8.2)

MEM ASSUME $\tilde{\phi}_{xx}[m]$ KNOWN $|m| \le P$

$c_{xx}[m]$

UNKNOWN FOR $|m| > P$

FIND $\tilde{P}_{xx}(\omega) = \text{DTFT}\{\tilde{\phi}_{xx}[m]\}$, $|m| \le P$

$\tilde{P}_{xx}(\omega)$ AS "SMOOTH" AS POSSIBLE

USE ENTROPY AS A MEASURE OF SMOOTHNESS

$$H = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log_2 \left(\tilde{P}_{xx}(\omega) \right) d\omega$$

LET $\tilde{\phi}_{xx}[m] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{P}_{xx}(e^{j\omega}) e^{j\omega m} d\omega$

$$\Rightarrow \tilde{P}_{xx}(\omega) = \sum_{l=-(N-1)}^{N-1} \tilde{\phi}_{xx}[l] e^{-j\omega l}$$

$N-1 = P$ $\quad \tilde{\phi}_{xx}[m] = c_{xx}[m]$, $|m| \le P$

FIND $\dfrac{\partial H}{\partial \tilde{\phi}_{xx}[m]} = 0$, SOLVE FOR $\tilde{\phi}_{xx}[m]$

$|m| > P$

CONSIDER $\dfrac{\partial \tilde{P}_{xx}(\omega)}{\partial \tilde{\phi}_{xx}[m]} = \dfrac{\partial}{\partial \hat{\phi}_{xx}[m]} \sum_{\ell=-P}^{P} \tilde{\phi}_{xx}[\ell] e^{-j\omega\ell} = e^{-j\omega m}$

$$\frac{\partial \log \tilde{P}_{xx}(\omega)}{\partial \tilde{\phi}_{xx}[m]} = \frac{1}{\tilde{P}_{xx}(\omega)} \cdot \frac{\partial \tilde{P}_{xx}(\omega)}{\partial \tilde{\phi}_{xx}[m]} = \frac{1}{\tilde{P}_{xx}(\omega)} e^{-j\omega m}$$

FOR $\overline{|m| > P}$, $\dfrac{\partial H}{\partial \hat{\phi}_{xx}[m]} = \dfrac{\partial}{\partial \tilde{\phi}_{xx}[m]} \underbrace{\dfrac{1}{2\pi}\int_{-\pi}^{\pi} \log(\hat{P}_{xx}(\omega))\, d\omega}_{H}$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\partial}{\partial \tilde{\phi}_{xx}[m]} \log(\tilde{P}_{xx}(\omega))\, d\omega$$

$$= \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{-j\omega m}}{\tilde{P}_{xx}(\omega)}\, d\omega}_{\mathrm{IDTFT}\left\{\frac{1}{\tilde{P}_{xx}(\omega)}\right\}} = 0, \quad \text{FOR } |m| > P$$

B.C. $\tilde{\phi}_{xx}[m]$, $\tilde{P}_{xx}(\omega)$ EVEN

LET $\psi_{xx}[m] = \mathrm{IDTFT}\left\{\dfrac{1}{\hat{P}_{xx}(\omega)}\right\}$

1. $\psi_{xx}[m]$ IS REAL, AND EVEN FUNCTION of m
2. $\psi_{xx}[m] = 0$ FOR $|m| > P$

$$\frac{1}{\hat{P}_{xx}(\omega)} = \hat{P}_{xx}^{-1}(\omega) = \sum_{m=-P}^{P} \psi_{xx}[m] e^{-j\omega m}$$

$$\hat{P}_{xx}(\omega) = \frac{1}{\sum_{m=-P}^{P} \psi_{xx}[m] e^{-j\omega m}}$$

USING ZT NOTATION

let $\hat{P}_{xx}(\omega) = \tilde{P}_{xx}(z)\Big|_{z=e^{j\omega}}$; $\tilde{P}_{xx}(z) = \dfrac{1}{\sum_{m=-p}^{p} \phi_{xx}[m]\, z^{-m}}$

BECAUSE $\phi_{xx}[m]$ IS REAL, EVEN,

$$\hat{P}_{xx}(\omega) = \frac{\sigma_x^2}{A(z)A(z^{-1})} \; ; \quad A(z) = 1 - \alpha_1 z^{-1} - \alpha_2 z^{-2} + \dots - \alpha_p z^{-p}$$
$$= 1 - \sum_{k=1}^{p} \alpha_k z^{-k}$$

$$\tilde{P}_{xx}(\omega) = \frac{\sigma_x^2}{(1-\alpha_1 z^{-1} - \alpha_2 z^{-2} + \dots - \alpha_p z^{-p})(1-\alpha_1 z - \alpha_2 z^{2} \dots - \alpha_p z^{p})}$$

$$P_{xx}(\omega) = \sigma_x^2 \cdot H(z)H(z^{-1})\Big|_{z=e^{j\omega}}$$
$$= \sigma_x^2 \cdot H(e^{j\omega})H(e^{-j\omega}) = \sigma_x^2\, H(e^{j\omega}) \cdot H^*(\omega) = \sigma_x^2 |H(e^{j\omega})|^2$$

CONSIDER

$$w[n] \longrightarrow \boxed{H(e^{j\omega})} \longrightarrow x[n]$$

WHITE σ_w^2

$$H(e^{j\omega}) = H(z)\Big|_{z=e^{j\omega}} = \frac{G}{1-\sum_{k=1}^{p} \alpha_k z^{-k}} = \frac{G}{A(z)}$$

PSD BY MODELING

THREE TYPES of MODELS

1. $H(z) = B(z) = \sum_{\ell=0}^{q} b_\ell z^{-\ell}$ (FIR) MOVING AVERAGE (MA) MODEL

ALL ZERO

ALL POLE 2. $H(z) = \dfrac{G}{A(z)} = \dfrac{G}{1-\sum_{k=1}^{p} \alpha_k z^{-k}}$ (IIR) AUTO REGRESSIVE (AR)

POLES + ZEROS 3. $H(z) = \dfrac{B(z)}{A(z)} = \dfrac{\sum_{\ell=0}^{q} b_\ell z^{-\ell}}{1-\sum_{k=1}^{p} \alpha_k z^{-k}}$ AUTO-REGRESSIVE MOVING AVERAGE (ARMA)

SOLUTION to MEM PSD ESTIMATION

$$w[n] \rightarrow \boxed{H(z)} \rightarrow x[n]$$

$$\frac{X(z)}{W(z)} = H(z) = \frac{G}{1-\sum_{k=1}^{p} \alpha_k z^{-k}} = \frac{G}{A(z)}$$

$$X(z)\left(1-\sum_{k=1}^{p} \alpha_k z^{-k}\right) = G\, W(z)$$

$$x[n] - \sum_{k=1}^{p} \alpha_k\, x[n-k] = G\, w[n]$$

$$x[n] = \sum_{k=1}^{p} \alpha_k\, x[n-k] + G\, w[n]$$

$$\phi_{xx}[m] = E\big[x[n]\,x[n+m]\big] = E\Big[x[n]\underbrace{\Big(\Big(\sum_{k=1}^{p} \alpha_k\, x[n+m-k]\Big) + G\,w[n+m]\Big)}_{x[n+m]}\Big]$$

$$\phi_{xx}[m] = \sum_{k=1}^{p} \alpha_k\, E\big[x[n]\,x[n+m-k]\big] + E\big[G\,x[n]\,w[n+m]\big]$$

$$= \sum_{k=1}^{p} \alpha_k\, \phi_{xx}[m-k] + G\,\delta[m]$$

YULE-WALKER EQ's

$$\begin{cases} \phi_{xx}[m] = \sum_{k=1}^{p} \alpha_k\, \phi_{xx}[m-k] + \sigma_w^2, & m=0 \\ \phi_{xx}[m] = \sum_{k=1}^{p} \alpha_k\, \phi_{xx}[m-k], & m \geq 1 \end{cases}$$

SOLVING THE YULE-WALKER EQNs, $m \geq 1$, $P = 4$

$$\phi_{xx}[m] = \sum_{k=1}^{P} \alpha_k \phi_{xx}[|m-k|], \quad \text{FOR } m \geq 1$$

$\phi_{xx}[m] \rightleftharpoons \phi[m]$

$m=1$ $\quad \alpha_1 \phi[0] + \alpha_2 \phi[1] + \alpha_3 \phi[2] + \alpha_4 \phi[3] = \phi[1]$

$m=2$ $\quad \alpha_1 \phi[1] + \alpha_2 \phi[0] + \alpha_3 \phi[1] + \alpha_4 \phi[2] = \phi[2]$

$m=3$ $\quad \alpha_1 \phi[2] + \alpha_2 \phi[1] + \alpha_3 \phi[0] + \alpha_4 \phi[1] = \phi[3]$

$m=4$ $\quad \alpha_1 \phi[3] + \alpha_2 \phi[2] + \alpha_3 \phi[1] + \alpha_4 \phi[0] = \phi[4]$

GIVEN $\{\phi[m]\}$
FIND $\{\alpha_k\}$

$$\underbrace{\begin{pmatrix} \phi[0] & \phi[1] & \phi[2] & \phi[3] \\ \phi[1] & \phi[0] & \phi[1] & \phi[2] \\ \phi[2] & \phi[1] & \phi[0] & \phi[1] \\ \phi[3] & \phi[2] & \phi[1] & \phi[0] \end{pmatrix}}_{\underline{\underline{R}}} \underbrace{\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix}}_{\underline{\alpha}} = \underbrace{\begin{pmatrix} \phi[1] \\ \phi[2] \\ \phi[3] \\ \phi[4] \end{pmatrix}}_{\underline{P}}$$

TOEPLITZ MATRIX

$$\underline{\underline{R}}\,\underline{\alpha} = \underline{P}$$

$$\cancel{\underline{\underline{R}}^{-1}\underline{\underline{R}}}\,\underline{\alpha} = \underline{\underline{R}}^{-1}\underline{P} \Rightarrow \boxed{\underline{\alpha} = \underline{\underline{R}}^{-1}\underline{P}}$$

WIENER-HOPF EQUATION

MATRIX INVERSION METHODS

$\underline{\underline{R}}$ UNCONSTRAINED GAUSSIAN ELIMINATION $O(P^3)$

$\underline{\underline{R}}$ HERMITIAN SYMMETRIC CHOLESKY DECOMPOSITION $O(P^2)$

$\underline{\underline{R}}$ TOEPLITZ LEVINSON-DURBIN RECURSION $O(P^2)$

INTRO TO LINEAR PREDICTION

WSS RP $x[n]$

KNOW $x[n-1], x[n-2], x[n-3], \ldots, x[n-p]$

ESTIMATE $\hat{x}[n]$ AS A LINEAR COMBINATION OF PREVIOUS SAMPLES

Let $\hat{x}[n] = \sum_{k=1}^{p} \alpha_k x[n-k]$

Let $e[n] = x[n] - \hat{x}[n]$

Find $\{\alpha_k\}$ THAT MINIMIZES $\mathcal{E}^2 = E\{e^2[n]\}$

$= E\{(x[n] - \hat{x}[n])^2\}$

MMSE SOLUTION

Let $\mathcal{E}^2 = E[(x[n] - \hat{x}[n])^2] = E\Big[\Big(x[n] - \underbrace{\sum_{k=1}^{p} \alpha_k x[n-k]}\Big)^2\Big]$

Let $\phi_{xx}[i,k] = \phi[i,k] \triangleq E[x[n-i]\, x[n-k]]$ $\hat{x}[n]$

Find $\{\alpha_k\}$ TO MIN $\mathcal{E}^2$

$$\frac{\partial \mathcal{E}^2}{\partial \alpha_k} = \frac{\partial}{\partial \alpha_k} E\Big[\Big(x[n] - \underbrace{\sum_{k=1}^{p} \alpha_k x[n-k]}\Big)^2\Big]$$

$$= E\Big[2\Big(x[n] - \sum_{\ell=1}^{p} \alpha_\ell x[n-\ell]\Big) x[n-k]\Big] = 0$$