10/23/24

SMOOTHING THE PERIODOGRAM; INTRO TO MAXIMUM ENTROPY SPECTRAL ESTIMATION

(OS FS 11.3-11.4, NOTES 5.5-5.7, 6.0-6.3; ROBINSON'S PAPER)

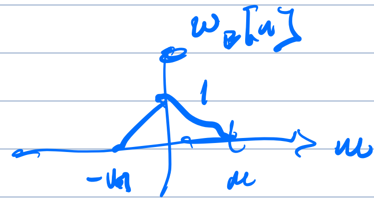
CLASSICAL PSD ESTIMATE

BIASED ACF EST. $C_{xx}[m] = \frac{1}{N} \sum_{n=0}^{N-|m|-1} x[n]\, x[n+m]$

HAVE $x[n]$ FOR $0 \le n \le N-1$

ESTIMATE $\phi_{xx}[m] = E\left[x[n]\, x[n+m]\right]$

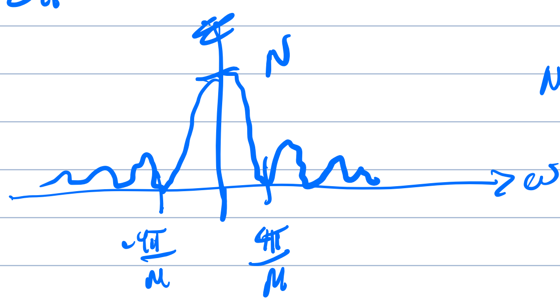
$$E\left[C_{xx}[m]\right] = \underbrace{\frac{N-|m|}{N}}_{w_B[m]} \phi_{xx}[m]$$



PERIODOGRAM

$$I_N(\omega) = \sum_{m=-(N-1)}^{N-1} C_{xx}[m]\, e^{-j\omega m}$$

$$E\left[I_N(\omega)\right] = \frac{1}{2\pi} W_B(e^{j\omega}) \circledast P_{xx}(\omega) = \frac{1}{2\pi}\int_{-\pi}^{\pi} W_B\left(e^{j(\omega-\theta)}\right) P_{xx}(\theta)\, d\theta$$



$M = N_W - 1$

VAR $\left[I_N(\omega)\right]$ DOES NOT $\rightarrow 0$ AS $N \rightarrow \infty$

Smoothing using Bartlett method

Given WSS RP $x[n]$, length N

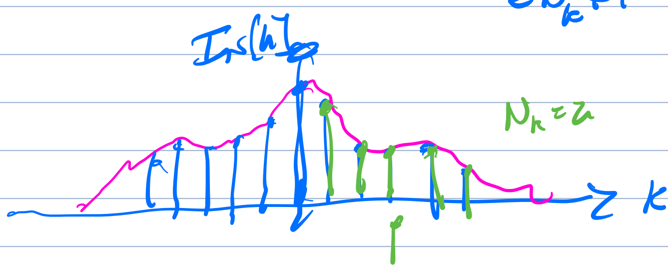
1. Break into K segments length M, $N = KM$
2. Compute periodogram for each segment
3. Average periodograms

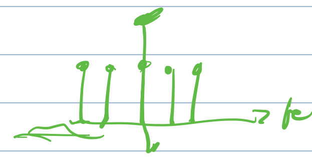
$$E[B_{xx}(\omega)] = \frac{1}{2\pi} P_{xx}(\omega) \circledast W_B(e^{j\omega})$$

Smoothing in frequency

Let $I_N[k] = I_N(\omega)\Big|_{\omega = 2\pi k}$

Let $$\hat{S}_{N_k}[k] = \frac{1}{2N_k+1} \sum_{l=k-N_k}^{k+N_k} I_N[k]$$





In continuous frequency

Let $$S_{xx}(\omega) = \frac{1}{2\pi} \int_{-\pi}^{\pi} I_N(\theta)\, W_s(e^{j(\omega-\theta)})\, d\theta = I_N(\omega) \circledast W_s(e^{j\omega})$$

In general want $W_s(e^{j\omega}) \geq 0, \ \forall \omega$

Let $w_s[m] \Longleftrightarrow W_s(e^{j\omega})$

$w_s[m]$ REAL, EVEN IN TIME

$$w_s[m] = \frac{1}{2\pi}\int_{-\pi}^{\pi} W_s(e^{j\omega})\, e^{j\omega m}\, d\omega$$

$$I_N(\omega) = \sum_{m=-(N-1)}^{N-1} C_{xx}[m]\, e^{-j\omega m}$$

$$= \frac{1}{N}\left|\sum_{n=0}^{N-1} x[n]\, e^{-j\omega n}\right|^2$$

$$S_{xx}(\omega) = \sum_{m=-(N-1)}^{N-1} C_{xx}[m]\, w_s[m]\, e^{-j\omega m}$$

$$= \frac{1}{N}\left|\sum_{n=0}^{N-1} x[n]\, w_s'[n]\, e^{-j\omega n}\right|^2$$

CHOOSE $w_s[m]$ TO BE AC FUNCTION of $w_s'[n]$

SO TO GET $S_{xx}[\omega]$

1. WINDOW DATA
2. COMPUTE $S_{xx}(\omega) = \frac{1}{N}\left|\sum_{n=0}^{N-1} x[n]\, w_s'[n]\, e^{-j\omega n}\right|^2$

WELCH METHOD (BLACKMAN - TUKEY METHOD)

Let $B_{xx}^{W}(\omega) = \frac{1}{K} \sum_{i=1}^{K} I_M^{(i)}(\omega)$

$$I_M(\omega) = \frac{1}{MU} \left| \sum_{n=0}^{M-1} x^{(i)}[n]\, w_s[n]\, e^{-j\omega n} \right|^2$$

$$U = \frac{1}{M} \sum_{n=0}^{M-1} w_s^2[n]$$

NEEDED FOR BIAS → 0

WELCH IS LIKE BARTLETT

1. $x^{(i)}[n]$ MODIFIED BY $w_s[n]$

2. $x^{(i)}[n]$ MAY OVERLAP A BIT

"MODERN" SPECTRAL ESTIMATION

(PSD BY MODELLING)

INTRO MAX ENTROPY PSD ESTIMATION

CLASSICAL PSD EST

1. ESTIMATE $\phi_{xx}[m]$ FOR $|m| \le P$ $(N-1)$
2. ESTIMATE P_{xx}, $\bar{\omega}$ $\phi_{xx}[m] = 0$, $|m| > P$

IN MEM:

1. MAKE NO ASSUMPTION ABOUT $\phi_{xx}[m]$ FOR $|m| > P$
2. FIND $\hat{P}_{xx}(\omega)$ THAT IS
   a. AS "SMOOTH" AS POSSIBLE
   b. INVERSE DTFT of $\hat{P}_{xx}(\omega) = \phi_{xx}[m]$, $|m| \le P$

USE ENTROPY AS A MEASURE of SMOOTHNESS

SHANNON INFO. DEF.

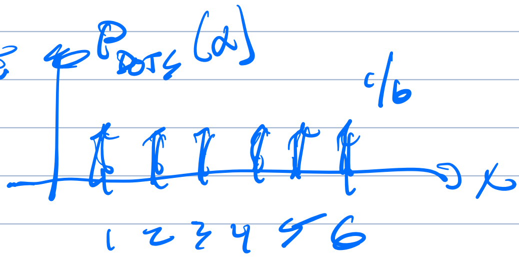
$$I(A_i) = -\log_2 P[A_i] = \log_2 \frac{1}{P[A_i]}$$

ENTROPY $H = E\left[I[A_i] \right]$

CONSIDER $\{A_i\}$, $\{P[A_i]\}$

$$H = \sum_{i=1}^{N} \underbrace{-\log_2(P[A_i])}_{I\{A_i\}} \cdot P[A_i]$$

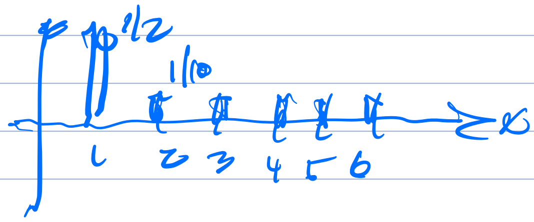
Ex. FAIR DICE



$$H = \sum_{i=1}^{6} -\log_2\left(\frac{1}{6}\right) \cdot \frac{1}{6}$$

$$= 2.585 \text{ BITS}$$

UNFAIR



$$H = -\log_2\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) + \sum_{i=1}^{5} -\log_2\left(\frac{1}{10}\right)\frac{1}{10}$$

$$= 2.161 \text{ BITS}$$

FOR PSD

let $H = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log_2 (P_{xx}(\omega))\, d\omega$

FIND PSD THAT MAXIMIZES H, $\hat{P}_{xx}(\omega)$

SUBJECT TO THE CONSTRAINT THAT $\frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{P}_{xx}(\omega) e^{j\omega m} d\omega = \hat{\Phi}_{xx}[m]$, FOR $|m| \le P$