10/21/24 ESTIMATING POWER SPECTRAL DENSITY FUNCTIONS

(OS 75: 11.0–11.4, NOTES 5.4–5.7)

ESTIMATES OF A.C. FUNCTION $\phi_{xx}[m] = E[x[n]\,x[n+m]]$

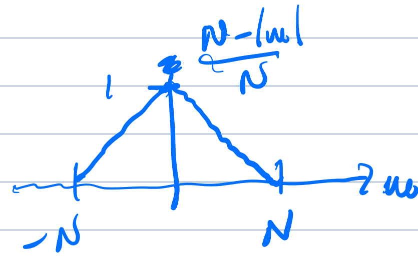
UNBIASED ESTIMATE, ASSUME THAT WE OBSERVE $x[n]$, $0 \le n \le N-1$

$$c'_{xx}[m] = \frac{1}{N-|m|} \sum_{n=0}^{N-|m|-1} x[n]\,x[n+m]$$

$$E[c'_{xx}[m]] = \phi_{xx}[m] \Rightarrow B = 0$$

$$c_{xx}[m] = \frac{1}{N} \sum_{n=0}^{N-|m|-1} x[n]\,x[n+m] = \frac{N-|m|}{N}\, c'_{xx}[m]$$

$$E[c_{xx}[m]] = \frac{N-|m|}{N} \cdot \phi_{xx}[m]$$



$$\mathrm{VAR}[c'_{xx}[m]] \simeq \frac{1}{(N-|m|)^2} \cdot \underbrace{\sum_{\ell=-\infty}^{\infty} \left(\phi_{xx}^2[\ell] + \phi_{xx}[\ell+m]\,\phi_{xx}[\ell-m] \right)}$$

$$\mathrm{VAR}[c_{xx}[m]] = \frac{\cancel{(N-|m|)^2}}{N^2} \cdot \frac{1}{\cancel{(N-|m|)^2}} \cdot \sum_{\ell}$$

PSD FUNCTIONS

WIENER-KHINTCHINE THM.

$$P_{xx}(\omega) = \sum_{m=-\infty}^{\infty} \phi_{xx}[m]\, e^{-j\omega m}$$

$$\phi_{xx}[m] = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_{xx}(\omega)\, e^{j\omega m}\, d\omega$$

$$\phi_{xx}[0] = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_{xx}(\omega)\, d\omega$$

IF $x[n]$ REAL, $\phi_{xx}[m]$ REAL, EVEN

$P_{xx}(\omega)$ REAL, EVEN, PERIODIC, PERIOD 2π

$P_{xx}(\omega) \geq 0, \ \forall \omega$

ESTIMATES OF PSD:

"REASONABLE" ESTIMATE: THE PERIODOGRAM

$$I_N(\omega) = \sum_{m=-(N-1)}^{N-1} c_{xx}[m]\, e^{-j\omega m}$$

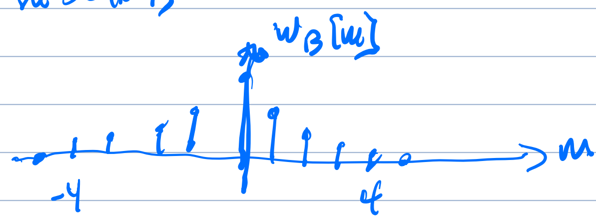
MEAN OF PERIODOGRAM

$$E[I_N(\omega)] = E\left[\sum_{m=-(N-1)}^{N-1} c_{xx}[m]\, e^{-j\omega m}\right]$$

$$= \sum_{m=-(N-1)}^{N-1} E\{c_{xx}[m]\}\, e^{-j\omega m}$$

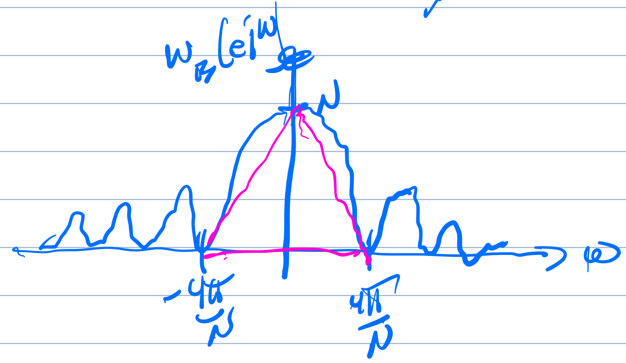
$$= \sum_{m=-(N-1)}^{N-1} \underbrace{\frac{N-|m|}{N}} \cdot \phi_{xx}[m]\, e^{-j\omega m}$$

$N=5$



$$E[I_N(\omega)] = \frac{1}{2\pi} FT[w_B[m]] \circledast P_{xx}[m]$$

$$= \frac{1}{2\pi}\,\frac{1}{N}\left(\frac{\sin\left(\frac{\omega N}{2}\right)}{\sin\left(\frac{\omega}{2}\right)}\right)^2 \circledast P_{xx}(\omega)$$



$$\lim_{N\to\infty} \frac{1}{2\pi} W_B(e^{j\omega}) = \delta(\omega)$$

$$\lim_{N\to\infty} E[I_N(\omega)] = P_{xx}(\omega)$$

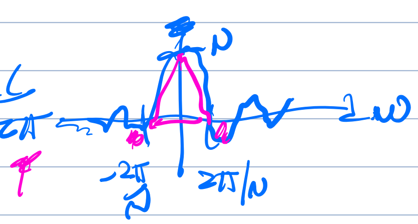
ALTERNATE EST. of PSD

$$I'_N(\omega) = \sum_{m=-(N-1)}^{N-1} c'_{xx}[m]\, e^{-j\omega m}$$

$$E[I'_N(\omega)] = \sum_{m=-(N-1)}^{N-1} E[c'_{xx}[m]]\, e^{-j\omega m} = \sum_{m=-(N-1)}^{(N-1)} \phi_{xx}[m]\, e^{-j\omega m}$$

$$= \sum_{m=-\infty}^{\infty} w_B[m] \cdot \phi_{xx}[m]\, e^{-j\omega m}$$

$$= \frac{1}{2\pi} W_B(e^{j\omega}) \circledast P_{xx}(\omega)$$

$= \frac{1}{2\pi}$  $\circledast P_{xx}(\omega)$

USE $I_N(\omega)$

BECAUSE $I_N(\omega) \geq 0, \forall \omega$

DON'T USE $I'_N(\omega)$!

COMMENT: $I_N(\omega) = \sum_{m=-(N-1)}^{N-1} C_{xx}[m] e^{-j\omega m}$

IF WE LET $X_N(e^{j\omega}) = \sum_{n=0}^{N-1} x[n] e^{-j\omega n}$

THEN $I_N(e^{j\omega}) = \frac{1}{N} |X_N(e^{j\omega})|^2$

VARIANCE OF PERIODOGRAM

CONSIDER RP $x[n]$, REAL, ZM, WHITE, GAUSSIAN, VARIANCE σ_x^2

FIND COVARIANCE OF $I_N(\omega)$

$$\underset{I_N}{\mathrm{Cov}}(\omega_1, \omega_2) = \overset{a}{E[I_N(\omega_1) I_N(\omega_2)]} - \overset{b}{E[I_N(\omega_1)] E[I_N(\omega_2)]}$$

USE $I_N(\omega) = \frac{1}{N} |X_N(e^{j\omega})|^2$

a $E[I_N(\omega_1) I_N(\omega_2)]$

$$I_N(\omega_1) = \underbrace{\left[\frac{1}{N} \sum_{\ell=0}^{N-1} x[\ell] e^{-j\omega_1 \ell} \cdot \sum_{m=0}^{N-1} x[m] e^{+j\omega_1 m}\right]}$$

$E[I_N(\omega_1) I_N(\omega_2)]$

$$= E\left[\frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{\ell=0}^{N-1} \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} x[k] x[\ell] x[m] x[n] e^{-j\omega_1(k-\ell)} e^{-j\omega_2(m-n)}\right]$$

$$= \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{\ell=0}^{N-1} \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} \overline{x[k] x[\ell] x[m] x[n]} e^{-j\omega_1(k-\ell)} e^{-j\omega_2(m-n)}$$

IF $x[n]$ GAUSSIAN, CAN APPLY MOMENT FACTORING THM

$$E[x_1 x_2 x_3 x_4] = \overline{x_1 x_2 x_3 x_4} = \overline{x_1 x_2}\,\overline{x_3 x_4} + \overline{x_1 x_3}\,\overline{x_2 x_4} + \overline{x_1 x_4}\,\overline{x_2 x_3}$$

$$E[x[k] x[\ell]] = \delta[k-\ell] \sigma_x^2 e^{-j\omega_1(k-\ell)}$$

$$\Rightarrow E[I_N(\omega_1) I_N(\omega_2)] = \sigma_x^4 \left\{ 1 + \left(\frac{\sin\left((\omega_1+\omega_2)\frac{N}{2}\right)}{N \sin\left((\omega_1+\omega_2)\frac{1}{2}\right)}\right)^2 + \left(\frac{\sin\left((\omega_1-\omega_2)\frac{N}{2}\right)}{N \left(\sin(\omega_1-\omega_2)\right)\frac{1}{2}}\right)^2 \right\}$$

$$\lim_{N\to\infty} E\left[I_N(\omega_1) I_N(\omega_2)\right] \neq 0 !!$$

HOW TO IMPROVE VARIANCE?

BARTLETT METHOD ... GIVEN WSS RP $x[n]$, $0 \le n \le N-1$

1. DIVIDE $x[n]$ INTO K ABUTTING SUBSEQUENCES OF LENGTH M
2. COMPUTE PERIODOGRAM FOR EACH SUBSEQUENCE
3. AVERAGE PERIODOGRAMS

$N = KM$ let $x^{(i)}[n] = x[n + (i-1)M]$

BARTLETT PSD $= B_{xx}(\omega) = \frac{1}{K} \sum_{i=1}^{K} I_M^{(i)}(\omega)$

$$E\left[B_{xx}(\omega)\right] = \frac{1}{K} E\left[\sum_{i=1}^{K} I_M^{(i)}(\omega)\right]$$

$$= \frac{1}{K} \sum_{i=1}^{K} E\left[I_M^{(i)}(\omega)\right] = E\left[I_M^{(i)}(\omega)\right]$$

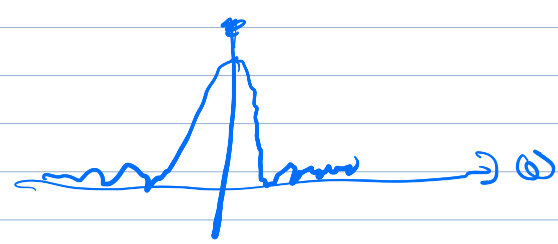
$$= \frac{1}{2\pi} \underbrace{W_B(e^{j\omega})}_{\text{LENGTH } M} \circledast P_{xx}(\omega)$$

AS K INCREASES, BIAS (CAUSED BY BLURRING) INCREASES

$$E\left[B_{xx}(\omega)\right] = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_{xx}(\theta)\, W_B\left(e^{j(\omega-\theta)}\right) d\theta$$

SMALL K

LARGER K



$$\mathrm{VAR}\left\{B_{xx}(\omega)\right\} = \frac{1}{K^2}\,\mathrm{VAR}\left[\sum_{i=1}^{K} I_M^{(i)}(\omega)\right]$$

$M = N/K$

$$= \frac{1}{K^2}\left[\sum_{i=1}^{K}\mathrm{VAR}\left\{I_M^{(i)}(\omega)\right\}\right]$$

$$= \frac{1}{K}\,\mathrm{VAR}\left[I_M^{(i)}(\omega)\right]$$

FIXED