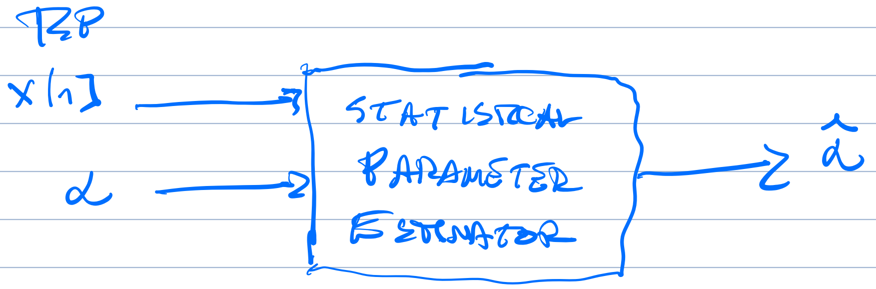
10/7/24 INTRODUCTION TO PARAMETER ESTIMATION

(LOS 75 CHAP 8.3–8.4, 11.0–11.2; NOTES 5.0–5.3)



ATTRIBUTES of "GOOD" ESTIMATES

$E[\hat{\alpha}] = m_{\hat{\alpha}}$

1. BIAS $B = \alpha - E[\hat{\alpha}]$ (WANT ZERO)

2. VARIANCE $E[(\hat{\alpha} - E[\hat{\alpha}])^2] = E[\hat{\alpha}^2] - m_{\hat{\alpha}}^2$

$= \sigma_{\hat{\alpha}}^2$ (WANT SMALL)

3. MEAN-SQUARED ERROR (MSE)

$E[(\hat{\alpha} - \alpha)^2] = \sigma_{\hat{\alpha}}^2 + B^2$ (WANT SMALL)

4. CONSISTENCY

AN ESTIMATE IS "CONSISTENT" IF

1. $\lim_{N \to \infty} B = 0$ 2. $\lim_{N \to \infty} \sigma_{\hat{\alpha}}^2 = 0$

N → # SAMPLES

~~SEE~~ $\{x[n]\}$ STAT INDEP

$m_x \neq 0$

EX. WSS ~~SHAPE~~ RP $x[n]$

Let $\hat{m}_x = \frac{1}{N} \sum_{n=0}^{N-1} x[n]$ SAMPLE MEAN

$\hat{\sigma}_{x}^2 = \frac{1}{N} \sum_{n=0}^{N-1} (x[n] - \hat{m}_x)^2$ SAMPLE VARIANCE

$$E[\hat{m}_x] = E\left[\frac{1}{N}\sum_{n=0}^{N-1} x[n]\right] = \frac{1}{N}\sum_{n=0}^{N-1} \underbrace{E[x[n]]}_{m_x} = m_x$$

$B = 0$

VARANCE of SAMPLE MEAN

$$\sigma^2_{\hat{m}_x} = E\left[\hat{m}_x^2\right] - m_x^2$$

$$= E\left[\frac{1}{N}\sum_{n=0}^{N-1} x[n] \cdot \frac{1}{N}\sum_{\ell=0}^{N-1} x[\ell]\right] - m_x^2$$

$$= \frac{1}{N^2}\sum_{n=0}^{N-1}\sum_{\ell=0}^{N-1} E\left[x[n]\,x[\ell]\right] - m_x^2$$

$E[x^2[n]] = \sigma_x^2 + m_x^2, \; n=\ell$

$= 0, \; n \neq \ell$

$$= \frac{1}{N^2}\sum_{n=0}^{N-1} x^2[n] + \frac{1}{N^2}\sum_{\substack{n=0\\ n\neq\ell}}^{N-1}\sum_{\ell=0}^{N-1} E[x[n]\,x[\ell]] - m_x^2$$

a b c

a $= \frac{1}{N^2}\sum_{n=0}^{N-1} \underbrace{E[x^2[\ell]]}_{\sigma_x^2 + m_x^2} = \frac{N}{N^2}(\sigma_x^2 + m_x^2) = \frac{1}{N}(\sigma_x^2 + m_x^2)$

b $= \frac{1}{N^2}\sum_{n=0}^{N-1}\sum_{\substack{\ell=0\\ \ell\neq N}}^{N-1} E[x[n]]\,E[x[\ell]] = \frac{1}{N^2}(N)(N-1)\, m_x^2 = \frac{N-1}{N}\, m_x^2$

c $= -m_x^2$

$$\sigma^2_{\hat{m}_x} = \frac{1}{N}(\sigma_x^2 + m_x^2) + \left(\frac{N-1}{N}\right) m_x^2 - m_x^2$$

$$= \frac{\sigma_x^2}{N} + \cancel{\frac{m_x^2}{N}} + \cancel{m_x^2} - \cancel{\frac{1}{N}m_x^2} - \cancel{m_x^2} = \frac{\sigma_x^2}{N}$$

$$= \frac{\sigma_x^2}{N}$$

$\hat{m}_x$ IS UNBIASED, CONSISTENT

MEAN OF SAMPLE VARIANCE

$$E\left[\hat{\sigma}_x^2\right] = E\left[\frac{1}{N}\sum_{n=0}^{N-1}\left(x[n]-\hat{m}_x\right)^2\right] = \frac{1}{N}\sum_{n=0}^{N-1} E\left[x^2[n] - 2\hat{m}_x x[n] + \hat{m}_x^2\right]$$

(a) (b) (c)

$$(a) = \frac{1}{N}\sum_{n=0}^{N-1} E\left[x^2[n]\right] = \sigma_x^2 + m_x^2$$

$$(b) = -\frac{2}{N^2}\sum_{n=0}^{N-1} E\left[\cancel{x}[n]\frac{1}{N}\sum_{\ell=0}^{N-1} x[\ell]\right] = -\frac{2}{N^2}\sum_{n=0}^{N-1}\sum_{\ell=0}^{N-1} E\left[x[n]\,x[\ell]\right]$$

$$= -\frac{2}{N^2}\sum_{n=0}^{N-1} E\left[x^2[n]\right] - \frac{2}{N^2}\sum_{n=0}^{N-1}\sum_{\substack{\ell=0\\ \ell\neq n}}^{N-1} E\left[x[n]\right]E\left[x[\ell]\right]$$

$$= -\frac{2}{N}\left(\sigma_x^2 + m_x^2\right) - \frac{2}{N^{\cancel{2}}}\cancel{N}(N-1)\,m_x^2$$

$$= -\frac{2}{N}\left(\sigma_x^2 + m_x^2\right) + \frac{2}{N}m_x^2 - 2m_x^2$$

$$(c) = E\left[\frac{1}{N}\sum_{n=0}^{N-1}\hat{m}_x^2\right] \qquad \sigma_{\hat{m}_x}^2 = E\left[\hat{m}_x^2\right] - m_x^2$$

$$E\left[\hat{m}_x^2\right] = \sigma_{\hat{m}_x}^2 + m_x^2$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} E\left[\hat{m}_x^2\right] = \frac{1}{N}\sum_{n=0}^{N-1}\left(\sigma_{\hat{m}_x}^2 + m_x^2\right) = \sigma_{\hat{m}_x}^2 + m_x^2$$

(a) + (b) + (c)

$$= \sigma_x^2 + m_x^2 - \frac{2}{N}\sigma_x^2 - \cancel{\frac{2}{N}m_x^2} + \cancel{\frac{2}{N}m_x^2} - 2m_x^2 + \sigma_{\hat{m}_x}^2 + m_x^2$$

$$= \sigma_x^2\left(1-\frac{2}{N}\right) + \cancel{\sigma_{\hat{m}_x}^2}\ \frac{\sigma_x^2}{N}$$

$$= \sigma_x^2\left(1-\frac{2}{N}+\frac{1}{N}\right) = \sigma_x^2\left(1-\frac{1}{N}\right) = \sigma_x^2\left(\frac{N-1}{N}\right)$$

BIASED

BUT $\lim_{N\to\infty} B_{\hat{\sigma}_x^2} = 0$

ESTIMATES OF AUTO-CORRELATION FUNCTIONS + POWER SPECTRA

ASSUME $x[n]$ ZM WSS RP, ERGODIC

$$\gamma_{xx}[m] = \phi_{xx}[m] = E\left[x[n]\,x[n+m]\right] = \langle x[n]\,x[n+m]\rangle$$

"REASONABLE" $\tilde{\phi}_{xx}[m]$

ASSUME WE ONLY OBSERVE $x[n]$ FOR $0 \le n \le N-1$

Let
$$c'_{xx}[m] = \frac{1}{N-|m|} \sum_{n=\max(0,-m)}^{\min(N-1,\,N-m-1)} x[n]\,x[n+m]$$

$$E\left[c'_{xx}[m]\right] = \frac{1}{N-|m|} \sum_{n=0}^{N-|m|-1} \underbrace{E\left[x[n]\,x[n+m]\right]}_{\phi_{xx}[m]} = \phi_{xx}[m]$$

$$c'_{xx}[m] \to \phi_{xx}[m]$$

$\Rightarrow c'_{xx}[m]$ UNBIASED

JENKINS + WATTS

$$\mathrm{VAR}\left[c'_{xx}[m]\right] \simeq \frac{N}{(N-|m|)^2} \cdot \sum_{\ell=-\infty}^{\infty} \left(\phi_{xx}^2[\ell] + \phi_{xx}[\ell+m]\,\phi_{xx}[\ell-m]\right)$$

ALT. ESTIMATE PROPOSED

Let
$$c_{xx}[m] = \frac{1}{N} \sum_{n=0}^{N-|m|-1} x[n]\,x[n+m] = \frac{N-|m|}{N}\, c'_{xx}[m]$$

$$E\left[c_{xx}[m]\right] = \frac{N-|m|}{N}\, E\left[c'_{xx}[m]\right] = \frac{N-|m|}{N}\, \phi_{xx}[m]$$

$c_{xx}[m]$ BIASED

BUT AS $N \to \infty$ $B \to 0$

$$\mathrm{VAR}\left[c_{xx}[m]\right] = \left(\frac{N-|m|}{N}\right)^2 \mathrm{VAR}\left[c'_{xx}[m]\right]$$

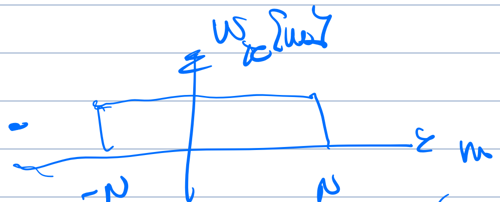
$$\mathrm{VAR}[c_{xx}[m]] \approx \frac{\cancel{(N-|m|)}^2}{N^2} \underbrace{\frac{1}{\cancel{(N-|m|)}^2} \cdot \sum_{\ell=-\infty}^{\infty}\left(\phi_{xx}^2[\ell] + \phi_{xx}[\ell+m]\,\phi_{xx}[\ell-m]\right)}_{\mathrm{VAR}[c'_{xx}[m]]}$$

$c_{xx}[m]$ BIASED, BIAS, VARIANCE $\to 0$ AS $N \to \infty$

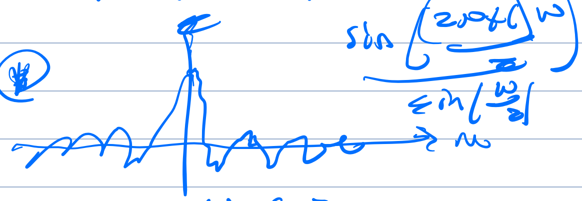
CONSISTENT

ANOTHER ISSUE

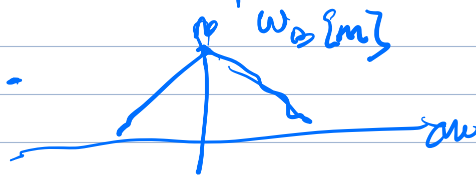
$$E[c'_{xx}[m]] = \phi_{xx}[m] \cdot w_R[m]$$



ESTIMATE PSD $\approx \frac{1}{2\pi} P_{xx}(\omega) \circledast W_R(e^{j\omega})$



$$E[c_{xx}[m]] = \phi_{xx}[m] \cdot w_B[m]$$



$$\Longleftrightarrow W_B(e^{j\omega}) \quad \frac{\sin^2\left[\frac{N\omega}{2}\right]}{\sin^2\left[\frac{\omega}{2}\right]}$$