10/2/24

Review of PSD, Random Processes Through Linear Filters, Intro to Parameter Estimation

[OS 75 8.3–8.7, 11.0–11.2; Notes 4.5–5.3]

Quiz 1 in one week, 10/9

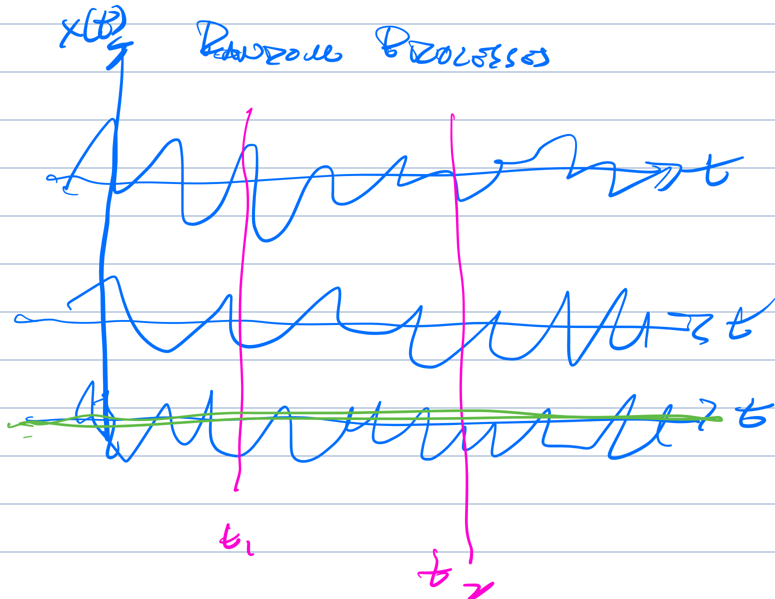
Coverage through PS 5; no RPs

Closed book, one sheet of notes

Friday review of Q1 '23

Sunday general review 2:30

Random Processes



If $E[x(t)] = \text{const.}$ m_x

$\phi_{xx}(t_1, t_2) = E[x(t_1)x(t_2)]$

depends only on $t_2 - t_1 = \tau$

then $x(t)$ WSS

$$E[g(x(t_1))] = \int_{-\infty}^{\infty} g(\alpha)\, p_{x(t_1)}(\alpha)\, d\alpha$$

If $x(t)$ stationary

CT $$\langle g(x(t)) \rangle = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} g(x(t))\, dt$$

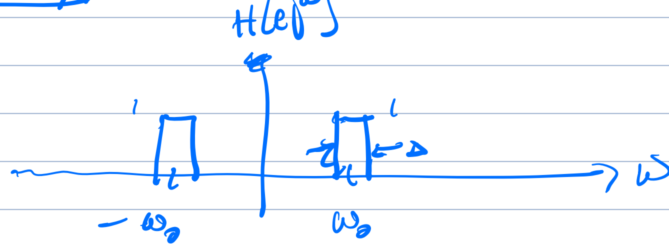
DT $$\langle g(x[n]) \rangle = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} g(x[n])$$

IF $E[g(x[n])] = \langle g(x[n]) \rangle$, RP ERGODIC

PHYSICAL DEF

$x[n] \rightarrow \boxed{h[n]} \rightarrow y[n]$

WSS $P_{xx}(\omega) = \sigma_x^2$



$$\lim_{\Delta \to 0} \text{ OUTPUT POWER } \{y[n]\} = \sigma_x^2 \cdot \frac{2\Delta}{2\pi} P_{xx}(\omega)\Big|_{\omega_0}$$

SAMPLE FN. DEF.

RP $x[n]$

$$x_N[n] = x[n], \quad |n| \le N$$
$$\qquad 0, \quad \text{ELSE}$$

$$P_{xx}(\omega) = \lim_{N \to \infty} \frac{1}{2N+1} E\left[\left| \sum_{n=-N}^{N} x_N[n] e^{-j\omega n} \right|^2 \right]$$

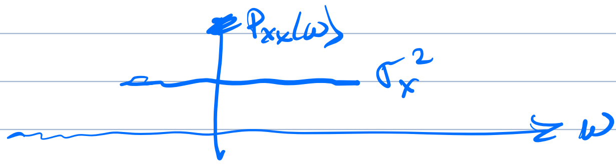
AUTO-CORRELATION FNc DEF

$$\text{Let } \phi_{xx}[m] = E[x[n]\, x[n+m]]$$

WIENER-KHINTCHINE THEOREM

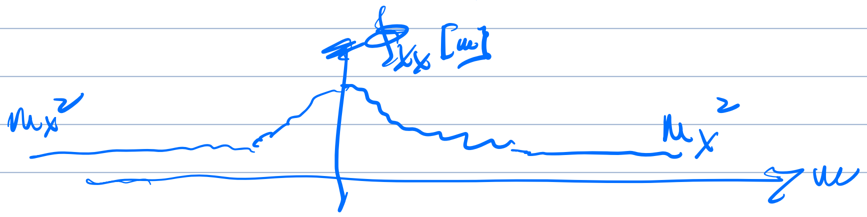
$$P_{xx}(\omega) = \sum_{m=-\infty}^{\infty} \phi_{xx}[m] e^{-j\omega m}$$

IF $P_{xx}(\omega)$ IS CONST. THEN RP $x[n]$ IS "WHITE"

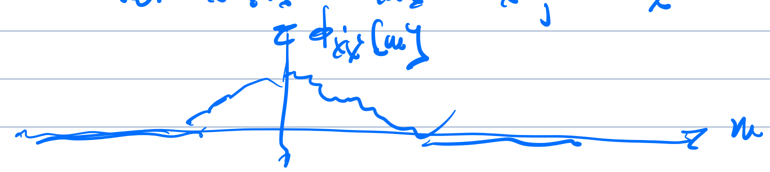


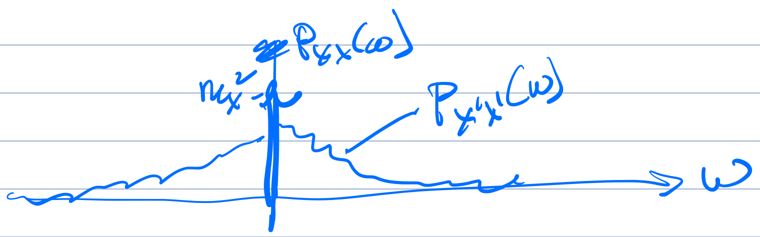
THEN $\phi_{xx}[m] = \sigma_x^2\, \delta[m]$

Consider $x[n]$, $m_x \neq 0$



Let $x'[n] = x[n] - m_x$; $m_{x'} = 0$





If $x[n]$ is white, $m_x = 0$

Consider $E\left[x[n]\,x[n+m]\right]$

For $m = 0$ $\quad E\left[x[n]\cdot x[n+m]\right] = \phi_{xx}[0] = \sigma_x^2$

For $m \neq 0$ $\quad E\left[x[n]\,x[n+m]\right] = \phi_{xx}[m] = 0$

$$E\left[x[n]\,x[n+m]\right] = E\left[x[n]\right]E\left[x[n+m]\right]$$

Uncorrelated; indep if Gaussian

Properties of autocorrelation fn

* If $x[n]$ real, $\phi_{xx}[m]$ even

  $x[n]$ complex, $\phi_{xx}[m]$ Hermitian symmetric

* If $x[n]$ is not periodic

  $\phi_{xx}[0] > \phi_{xx}[m]$

* $\lim_{|m| \to \infty} \phi_{xx}[m] = m_x^2$

Properties of PSD Functions

- If $x[n]$ real, $P_{xx}(\omega)$ real, even
- $P_{xx}(\omega) \geq 0$
- Total power $E[x^2[n]] = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_{xx}(\omega)\, d\omega$
- If $m_x = E[x[n]] \neq 0$, $P_{xx}(\omega)$ will have a term $\propto \delta(\omega)$

Random Processes through Linear Filters

$$\text{WSS } x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

Know m_x, $\phi_{xx}[m]$, $\gamma_{xx}[m]$, $P_{xx}(\omega)$, $h[n]$

Find m_y, $\phi_{yy}[m]$, $P_{yy}(\omega)$

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

$$m_y = E[y[n]] = E\left[\sum_k h[k]\, x[n-k]\right] = \sum_{k=-\infty}^{\infty} h[k]\, \underbrace{E[x[n-k]]}_{m_x}$$

$$\boxed{m_y = m_x \sum_{k=-\infty}^{\infty} h[k]}$$

$$\phi_{yy}[m] = E\left[y[n]\, y[n+m]\right]$$

$$= E\left[\underbrace{\sum_{k=-\infty}^{\infty} h[k]\, x[n-k]}_{y[n]}\ \underbrace{\sum_{l=-\infty}^{\infty} h[l]\, x[n+m-l]}_{y[n+m]}\right]$$

$$\phi_{yy}[m] = \sum_{k=-\infty}^{\infty} h[k] \sum_{l=-\infty}^{\infty} h[l]\, \underbrace{E\left[x[n-k]\, x[n+m-l]\right]}_{\phi_{xx}[m+k-l]}$$

Let $r = l-k$, $l = r+k$

$$\phi_{yy}[m] = \sum_{k=-\infty}^{\infty} h[k] \sum_{r=-\infty}^{\infty} h[r+k]\, \phi_{xx}[m-r]$$

$$= \sum_{r=-\infty}^{\infty} \phi_{xx}[m-r] \underbrace{\sum_{k=-\infty}^{\infty} h[k]\, h[k+r]}_{\phi_{hh}[r] = h[r] * h[-r]}$$

$$= \sum_{r=-\infty}^{\infty} \phi_{xx}[m-r]\, \phi_{hh}[r] = \phi_{hh}[m] * \phi_{xx}[m]$$

$$\boxed{\phi_{yy}[m] = \phi_{xx}[m] * \phi_{hh}[m] = \phi_{xx}[m] * h[m] * h[-m]}$$

$$\Downarrow$$

$$\boxed{P_{yy}(\omega) = P_{xx}(\omega) \cdot H(e^{j\omega}) \cdot H^*(e^{j\omega}) = P_{xx}(\omega) \left|H(e^{j\omega})\right|^2}$$

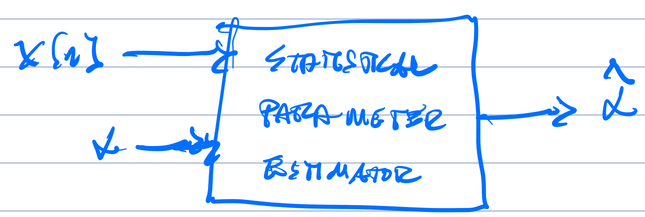
For $h[n]$ deterministic.

$$\sum_{n=-\infty}^{\infty} |h[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |H(e^{j\omega})|^2\, d\omega$$

INTRO TO PARAMETER ESTIMATION

WSS $x[n]$

Random parameter α



CHARACTERISTICS of "GOOD" ESTIMATES

1. BIAS $\quad B = \alpha - E[\hat{\alpha}] \quad$ (WANT $B = 0$)

2. VARIANCE $\quad E\left[\left(\tilde{\alpha} - E(\hat{\alpha})\right)^2\right] = \sigma_{\tilde{\alpha}}^2 = E[\alpha_{\tilde{x}}^2] - M_{\tilde{\alpha}}^2$

3. MEAN-SQUARE ERROR (MSE) $\quad E\left[\left[\tilde{\alpha} - \alpha\right]^2\right] = \sigma_{\tilde{\alpha}}^2 + B^2$

4. CONSISTENCY $\quad \tilde{\alpha}$ IS CONSISTENT IF

CONSIDER N SAMPLES of $x(n)$

$$\lim_{N \to \infty} B = 0 \quad , \quad \lim_{N \to \infty} \sigma_{\tilde{\alpha}}^2 = 0$$