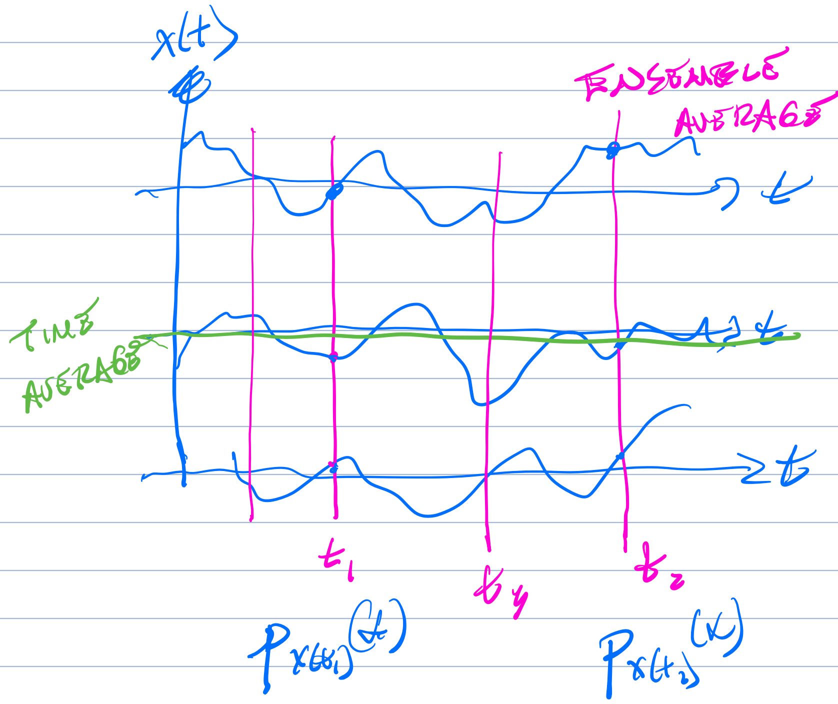
9/30/24 Properties of Random Processes

Ergodicity, Gaussianity, Power Spectral Density (PSD) Functions,
RP through LSI Systems

(OS 75 8.2-8.4, OS 4P Appendix A, Notes 4.3-4.5)

Quiz 1 10/4

* through PS 5, no prob/R.P.
* Review of Q1 '23 on Friday
* Review session Sunday 2/2:30 PM
* Closed book one sheet of notes



Statistical averages

$$E[g(x(t))] = \int_{-\infty}^{\infty} g(\alpha)\, p_{x(t_1)}(\alpha)\, d\alpha$$

Mean $E[x(t_1)] = \int_{-\infty}^{\infty} \alpha\, p_{x(t_1)}(\alpha)\, d\alpha$

$x(t_1) = X_{t_1}$ $\quad = m_{x(t_1)} = \mu_{x(t_1)}$

Variance $\sigma_x^2 = E[(x - m_x)^2]$

$$= E[x^2(t)] - (E[x(t)])^2$$

Autocorrelation Fn

$$\phi_{xx}[n_1, n_2] = R_{xx}(n_1, n_2) = E[x[n_1]\, x^*[n_2]]$$

Cross-correlation

$$\phi_{xy}[n_1, n_2] = E[x[n_1]\, y^*[n_2]]$$

Auto-covariance $\gamma_{xx}[n_1, n_2] = E[(x[n_1] - m_{x[n_1]})(x[n_2] - m_{x[n_2]})^*]$

$$\gamma_{xx}[n_1,n_2] = \underbrace{E\big[x[n_1]\,x[n_2]\big]}_{\phi_{xx}[n_1,n_2]} - m_{x[n_1]}\, m_{x[n_2]}$$

STATIONARITY

RP $x[n]$ IS WIDE-SENSE STATIONARY (WSS)

1. $m_{x[n]}$ IS CONSTANT m_x
2. $\phi_{xx}[n_1,n_2]$ DEPENDS ONLY ON $n_2-n_1=m$

$$\phi_{xx}[m]$$

IF $x[n]$ IS STATIONARY CAN DEFINE TIME AVERAGES

TIME AVERAGE DT

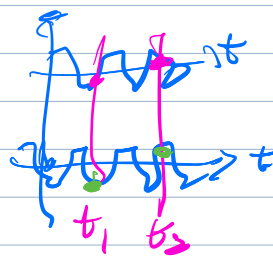
$$\langle g(x[n])\rangle = \lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{N} g(x[n])$$

CT

$$\langle g(x(t))\rangle = \lim_{T\to\infty}\frac{C}{2T}\int_{-T}^{T} g(x(t))\,dt$$

EX:

$$x(t) = A\cos(\omega t+\phi) \qquad p_\phi(\alpha) = \frac{1}{2\pi},\ 0\le\alpha\le 2\pi$$



$$E\{x(t)\}=0$$

$$E[x(t_1)\,x(t_2)] = \int_0^{2\pi} E\big[x(t_1)\,x(t_2)\,\big|\,\phi=\alpha\big]\,p_\phi(\alpha)\,d\alpha$$

$$= \int_0^{2\pi} A\cos(\omega t_1+\alpha)\,A\cos(\omega t_2+\alpha)\,\frac{1}{2\pi}\,d\alpha$$

$$= \frac{A^2}{2\pi}\cancel{\int_0^{2\pi}\frac{1}{2}\cos(\omega(t_1+t_2)+2\alpha)\,d\alpha} + \frac{A^2}{2\pi}\int_0^{2\pi}\frac{1}{2}\cos(\omega(t_1-t_2))\,d\alpha$$

$$+\frac{A^2}{(2)(\cancel{2\pi})}\cos(\omega(t_1-t_2))\,\cancel{2\pi}$$

$$=\frac{A^2}{2}\cos[\omega(t_2-t_1)]$$

$$=\frac{A^2}{2}\cos(\omega\tau)$$

$$\langle x(t)\rangle = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}A\cos(\omega t+\phi)\,dt = 0$$

$$\langle x(t)\,x(t+\tau)\rangle$$

$$=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}A\cos(\omega t+\phi)\,A\cos(\omega(t+\tau)+\phi)\,dt$$

$$=\cancel{\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\frac{A^2}{2}\cos(2\omega t+\omega\tau+2\phi)\,dt}$$

$$+\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\frac{A^2}{2}\cos(\omega\tau)\,dt$$

$$\lim_{T\to\infty}\frac{1}{\cancel{2T}}\underbrace{\frac{A^2}{2}\cos(\omega\tau)}\,\cancel{2T}$$

$$E[x(t)] = \langle x(t)\rangle = 0$$

$$E[x(t)\,x(t+\tau)] = \langle x(t)\,x(t+\tau)\rangle = \frac{A^2}{2}\cos(\omega\tau)$$

$x(t)$ IS WIDE SENSE ERGODIC

GAUSSIAN RANDOM PROCESSES

RP $x[n]$ IS GAUSSIAN IF

$$p_{x[n_1]x[n_2]x[n_3]\ldots x[n_N]}(\alpha, \beta, \gamma, \delta, \ldots)$$

IS GAUSSIAN

GRV $\quad p_{x[n_1]}(\alpha) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(\alpha - m_x)^2}{2\sigma^2}}$

RP $x[\underline{n}]$

$$= p_{x[\underline{n}]}(\underline{\alpha}) = \frac{1}{(2\pi)^{N/2} |\underline{\underline{C_x}}|^{\frac{1}{2}}} e^{-\frac{1}{2}(\underline{\alpha} - \underline{m_x})^T \underline{\underline{C_{xx}^{-1}}} (\underline{\alpha} - \underline{m_x})}$$

$1 \times N \quad N \times N \quad N \times 1$

PROPERTIES

1. GRP + WSS => SSS!
2. LINEAR TRANSFORMATIONS of GRPs => GRPs

POWER SPECTRAL DENSITY FUNCTIONS (PSD)

FOR DETERMINISTIC TIME FUNCTIONS,

$$\text{ENERGY} = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$$

ENERGY DENSITY FUNCTION

$$\text{POWER} = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

WSS RP

AC $\phi_{xx}[m] = E\{x[n]\,x[n+m]\}$

AUTO-COV $\gamma_{xx}[m] = E\{x[n]\,x[n+m]\} - m_{x[n]}\,m_{x[n+m]}$ (m_x^2)

FOR $m=0$, $\phi_{xx}[0] = E[x^2[n]]$

$\gamma_{xx}[0] = E[x^2[n]] - m_x^2 = \sigma_x^2$

$\sigma_x^2 = E[x^2[n]] - m_x^2$

$E[x^2[n]] = \sigma_x^2 + m_x^2$

TOTAL POWER, AC POWER, DC POWER

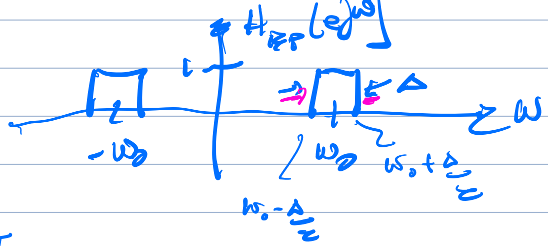
POWER SPECTRAL DENSITY FUNCTION

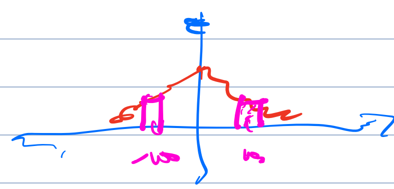
1. PHYSICAL DEF
2. SAMPLE FUNCTION DEF
3. AUTO CORRELATION DEF

PHYSICAL DEF

ZM ZERO MEAN WSS $x[n]$ RP

$x[n] \rightarrow$ BPF $\rightarrow$ MEASURE POWER OUT





$$\sigma_x^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_{xx}(\omega)\,d\omega$$

POWER SPECTRAL DENSITY FXC

LET $\lim_{\Delta \to 0}$ OF POWER OUTPUT OF FILTER $= \sigma_x^2 = \frac{2\Delta}{2\pi} P_{xx}(\omega)\Big|_{\omega=\omega_0}$

WSS RP

SAMPLE

FN. DEFINITION

CONSIDER $x[n]$

Let $x_N[n] = \begin{cases} x[n], & n \leq N \\ 0, & \text{ELSE} \end{cases}$

$$P_{xx}(\omega) = \lim_{N \to \infty} \frac{1}{2N+1} E\left[\left| \sum_{n=-N}^{N} x_N[n] e^{-j\omega n} \right|^2 \right]$$

AUTO. CORRELATION DEF

1. ESTIMATE $\phi_{xx}[m] = E\left[x[n]\, x[n+m] \right]$

2. CALCULATE DTFT $\{ \phi_{xx}[m] \}$

WIENER-KHINTCHIN THEOREM

3. $P_{xx}(\omega) = \sum_{m=-\infty}^{\infty} \phi_{xx}[m] e^{-j\omega m}$