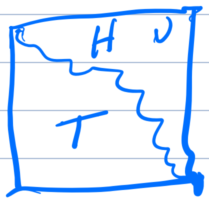
9/25/24 INTRO TO PROBABILITY & RANDOM PROCESSES

(APPENDIX A IN OSUP, OS 75 CHAPTER 8, NOTES 4.1-4.3)

& DRAKE BOOK

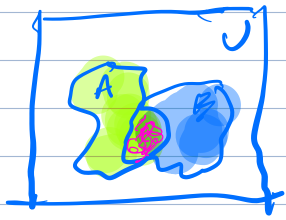


EACH EVENT A_i HAS PROBABILITY P_i

RULES OF PROBABILITY

1. $0 \le P_i \le 1$

2. $\sum_{i=1}^{N} P[A_i] = 1$



CONDITIONAL PROBABILITY

$$P[A|B] = \frac{P[AB]}{P[B]} \qquad P[A \cap B]$$

$$P[A+B] = P[A \cup B] = P[A] + P[B] - P[AB]$$

$$P[A|B] = \frac{P[AB]}{P[B]}, \qquad P[B|A] = \frac{P[AB]}{P[A]}$$

$$P[AB] = P[A|B]\,P[B] = P[B|A]\,P[A]$$

$$P[A|B] = \frac{P[B|A]\,P[A]}{P[B]}$$ BAYES RULE!

RANDOM VARIABLE X

OUTCOME of A RANDOM EXP. FOR WHICH OUTCOME IS NUMERICAL

$$p_x(\alpha)$$

$$P[a < x \le b] = \int_a^b p_x(\alpha)\,d\alpha$$

$$p_x(\alpha) \ge 0$$

$$\int_{-\infty}^{\infty} p_x(\alpha)\,d\alpha = 1$$

JOINT PDFs of 2 RVs .... RV X, Y $\quad p_{xy}(\alpha, \beta)$

$$P[a_1 < x \le a_2,\ b_1 < y \le b_2] = \int_{b_1}^{b_2}\int_{a_1}^{a_2} p_{xy}(\alpha, \beta)\,d\alpha\,d\beta$$

$$\int_{-b_1}^{b_2} p_{xy}(\alpha, \beta)\,d\beta = p_x(\alpha)$$

$$p_{x|y}(\alpha|\beta) = \frac{p_{xy}(\alpha, \beta)}{p_y(\beta)}$$

STATISTICAL AVERAGE

RV X

$g(\cdot)$ DETERMINISTIC

$$E[g(x)] = \int_{-\infty}^{\infty} g(\alpha)\,p_x(\alpha)\,d\alpha$$

EXAMPLES ...

1. $g(x) = x;\quad E[g(x)] = E[x] = \int_{-\infty}^{\infty} \alpha\,p_x(\alpha)\,d\alpha = m_x = \mu_x$

2. VARIANCE

$$\sigma_x^2 = E[(x - m_x)^2] = E[x^2] - m_x^2$$

STANDARD DEVIATION $\sigma_x = \sqrt{\sigma_x^2}$

3. COVARIANCE (x, y)

$$\gamma(x,y) = E[(x - m_x)(y - m_y)] = E[xy] - m_x m_y$$

CORRELATION $(x, y) = \phi(x,y) = E[xy]$

$\gamma(x,y) = 0$ IF x, y STAT. INDEP.

RANDOM VARIABLES x, y ARE STATISTICALLY INDEPENDENT

IF $P[x|y] = P[x]$

$$P[x|y] = \frac{P[xy]}{P[y]} = P[x]$$

IF x, y STAT INDEP, $P[xy] = P[x]\,P[y]$

$$\Rightarrow \quad p_{xy}(\alpha, \beta) = p_x(\alpha) \cdot p_y(\beta)$$

IF x, y INDEP, $E[xy] = E[x]\,E[y] = E[g(x, y)] = E[g(x)]$

GIVEN RV x, m_x, σ_x^2

$$E[ax] = a\, m_x$$

$$\text{VAR}[ax] = a^2 \sigma_x^2$$

GIVEN RVs, x, y

1. $E[x + y] = E[x] + E[y]$ ALWAYS!
2. $\text{VAR}[x + y] = \text{VAR}[x] + \text{VAR}[y]$ IFF x, y ARE STAT. INDEP.
3. $p_{x+y}(\alpha) = p_x(\alpha) * p_y(\alpha)$ IFF x, y STAT. INDEP.

GIVEN SET OF EVENTS $\{A_i\}_{i=1}^{n}$, $E[x] = \sum_{i=1}^{n} E[x|A_i] \cdot P[A_i]$

RANDOM PROCESSES   RP x AT TIME n_1

$x[n_1]$ OR x_{n_1}

STATIONARITY

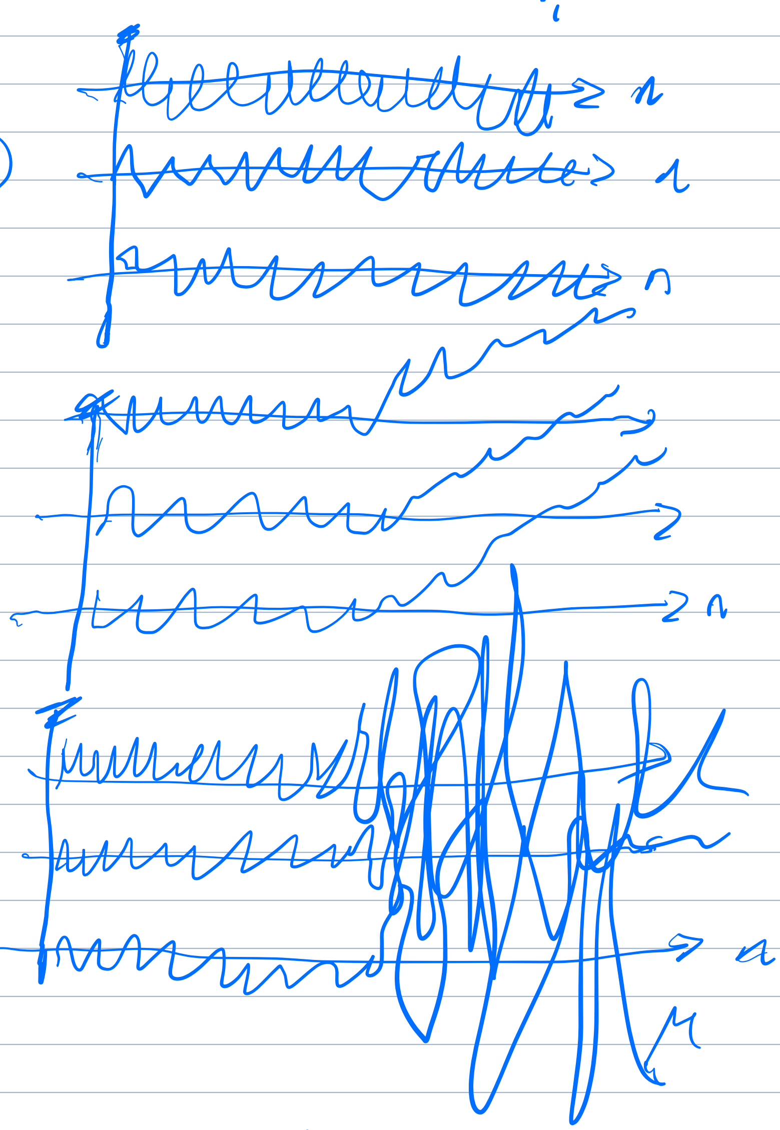
CONSIDER:

STATIONARITY

1

2

3



DEFS of STATIONARITY

STRICT-SENSE STATIONARITY (SSS)

1. (MEAN) $E\{x[n]\} = \text{CONSTANT} = E\{x\} = m_x$ FOR ALL n
2. (AC) $\phi_{xx}[n_1, n_2] = E\left[x[n_1]\, x^*[n_2]\right]$ MUST DEPEND ONLY ON $n_2 - n_1 \triangleq m$
3. FOR ANY COLLECTION of TIMES $n_1, n_2, n_3, \ldots$

$$E\left[x[n_1]\, x[n_2]\, x[n_3]\, x[n_4] \ldots\right]$$

MUST DEPEND ONLY ON $n_2 - n_1,\ n_3 - n_1,\ n_3 - n_2,\ n_4 - n_1, \ldots$

WIDE-SENSE STATIONARITY (WSS)

1st TWO REQs ABOVE

1. (MEAN) $E[x[n]] = \text{CONSTANT} = E[x] = m_x$ FOR ALL n

2. (AC) $\phi_{xx}[n_1, n_2] = E[x[n_1]\, x^*[n_2]]$ MUST DEPEND ONLY ON $n_2 - n_1 \triangleq m$

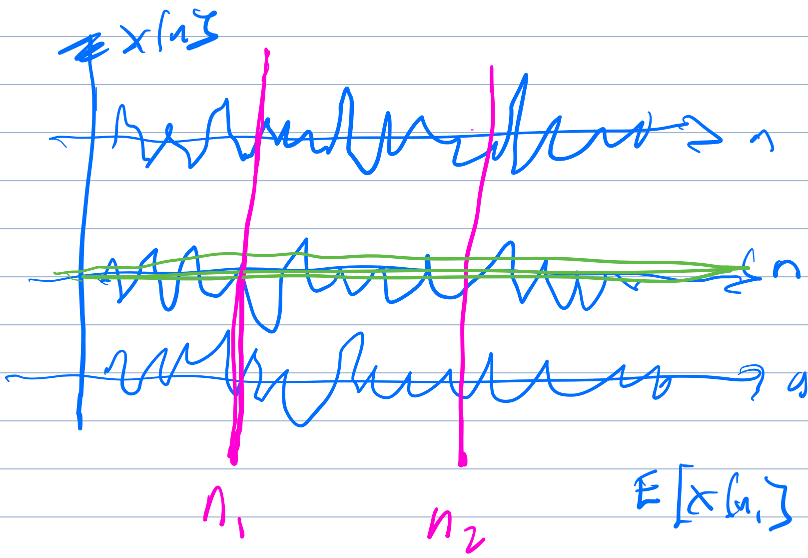
$E[x[n]] = E[x] = m_x$

$$\phi_{xx}[n_1, n_2] = E[x[n_1]\, x[n_2]] = \phi_{xx}[n_2 - n_1] = \phi_{xx}[m]$$

SECOND-ORDER SPEC OF STATIONARITY

IF A RP IS WSS + GAUSSIAN, IT IS ALSO SSS

TIME AVERAGES OF AN RP



$E[x[n_1]\, x[n_2]]$

IF $x[n]$ STATIONARY

DEFINE TIME AVERAGE of $g[x[n]]$

DISCRETE-TIME

$$\langle g[x[n]] \rangle = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} g(x[n])$$

CONTINUOUS-TIME

$$\langle g(x(t)) \rangle = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} g(x(t))\, dt$$

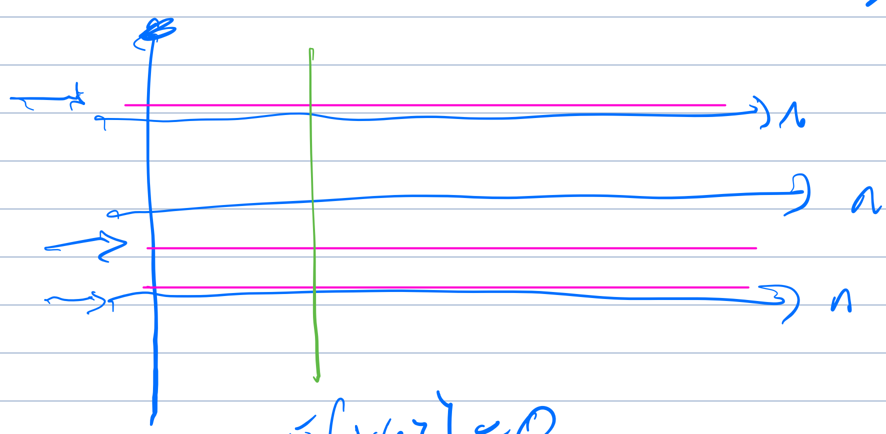
IF $E[g(x[n])] = \langle g(x[n]) \rangle$ FOR ALL $g(\cdot)$

THEN RP IS ERGODIC

EX. RP $x[n] = K$, K pdf $\mathcal{N}(0,1)$

STATIONARY

NOT ERGODIC



$E[x[n]] = 0$