9/16/24

INTRO TO SHORT-TIME FOURIER TRANSFORMS (STFTs)

(WO 6.0–6.2, NOTES 3.1–3.2)

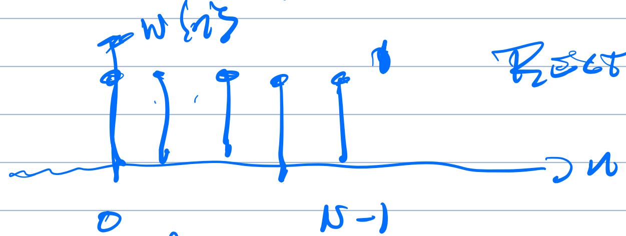
DTFT $x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}\, d\omega$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

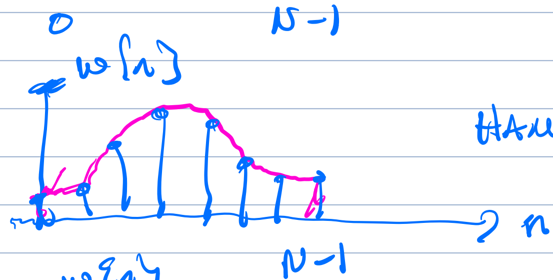
STFT:

$$X[n,\omega) = \sum_{m=-\infty}^{\infty} x[m]\, w[n-m]\, e^{-j\omega m}$$

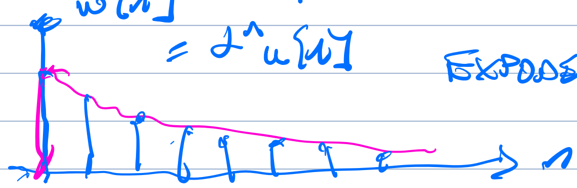
Types of windows:



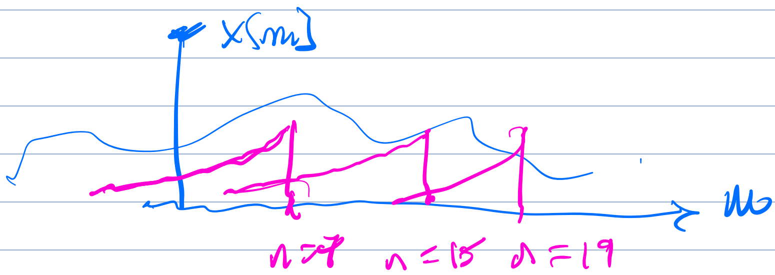
RECTANGULAR



HAMMING



EXPONENTIAL



DEMO: FT IMPLEMENTATION

$$\begin{tikzcd} x[m] \arrow[r] & \otimes \arrow[r] & \boxed{DTFT} \arrow[r] & X[n,\omega) \\ & w[n-m] \arrow[u] & & \end{tikzcd}$$

STFT

$\boxed{\text{FOURIER TRANSFORM IMPLEMENTATION}}$

Weinstein OPPENHEIM

$$X[n,\omega) = \sum_m x[m]\, w[n-m]\, e^{-j\omega m}$$

OPPENHEIM + SCHAFER

$$X[n,\omega) = \sum_m x[m+n]\, w[m]\, e^{-j\omega m}$$

WHAT IS ~~THE~~ WINDOWING OPERATION?

$$X[n,\omega) = \sum_m \left(x[m]\, w[n-m] \right) e^{-j\omega m}$$

DTFT of $w[n-m]$

$$w[n-m] \Longleftrightarrow \sum_{m=-\infty}^{\infty} w[n-m]\, e^{-j\omega m}$$

let $l = n-m$

$m = n-l$

$$= \sum_{l=-\infty}^{\infty} w[l]\, e^{-j\omega(n-l)}$$

$$= e^{-j\omega n} \underbrace{\sum_l w[l]\, e^{j\omega l}}_{W(e^{j\omega})}$$

$$X[n,\omega) = \sum_m x[m]\, w[n-m]\, e^{-j\omega m}$$

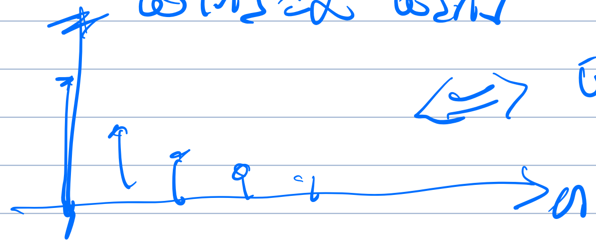
$$x[m]\, w[n-m] \Longleftrightarrow X(e^{j\omega}) * \bar{W}(e^{j\omega})\, e^{-j\omega n}$$

$$x[m]\,w[n-m] \Longleftrightarrow \frac{1}{2\pi}\int_{2\pi} X(e^{j(\omega-\theta)})\, W(e^{j\theta})\, e^{-j\theta}\, d\theta$$

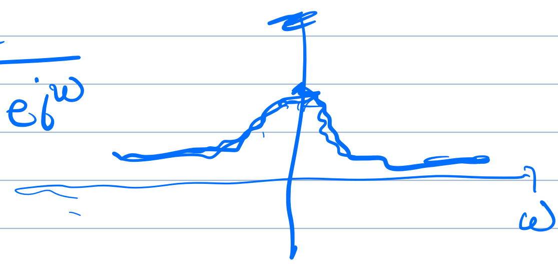
DFTs & POPULAR WINDOWS

EXPONENTIAL WINDOW

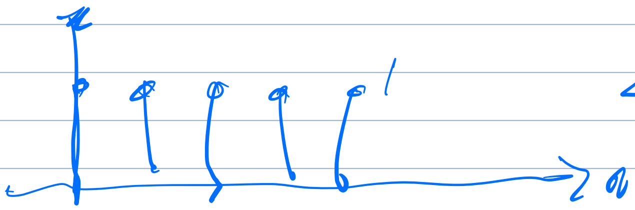
$w[n] = \alpha^n u[n]$



$$\Longleftrightarrow \quad W(e^{j\omega}) = \frac{1}{1-\alpha e^{j\omega}}$$



RECTANGULAR WINDOW



$$\Longleftrightarrow \quad e^{-j\omega\frac{(N-1)}{2}} \frac{\sin\left(\frac{\omega N}{2}\right)}{\sin\left(\frac{\omega}{2}\right)}$$

$W(e^{j\omega})$

STFT

$$X[n,\omega] = \sum_{m=-\infty}^{\infty} x[m]\,w[n-m]\,e^{-j\omega m}$$

DFT $\qquad$ DTFT

$$X[k] = X(e^{j\omega})\Big|_{\omega=\frac{2\pi k}{N}} = \omega_k, \quad 0 \le k \le N-1$$

$$X[n,k] = X[n,\omega]\Big|_{\omega=\omega_k=\frac{2\pi k}{N}} = \sum_{m=n-(N-1)}^{n} x[m]\,w[n-m]\,e^{-j\omega_k m}$$

$$e^{-j\frac{2\pi k}{N} m}$$

DEMO: WINDOW DURATION

ALTERNATE IMPLEMENTATIONS OF STFT

$$X[n,k] = \sum_{m=-\infty}^{\infty} \left(w[n-m]\, x[m] \right) e^{-j\omega_k m} \qquad \omega_k = \frac{2\pi k}{N}$$

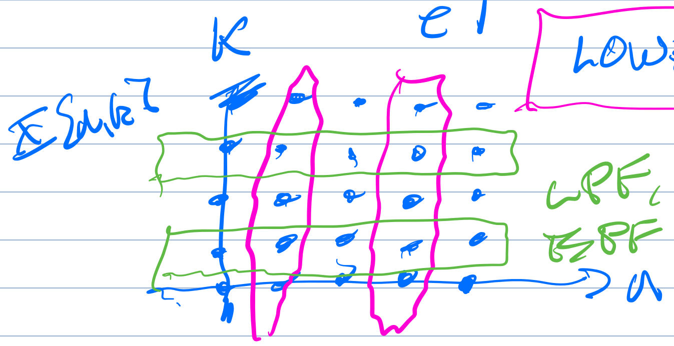
$$X[n,k] = \sum_{m=-\infty}^{\infty} w[n-m] \left(x[m] e^{-j\omega_k m} \right)$$

$$w[n] * x[n] e^{-j\omega_k n}$$

$$x[n] \longrightarrow \otimes \longrightarrow \boxed{w[n]} \longrightarrow X[n,k]$$

$$e^{-j\omega_k n}$$

LOWPASS FILTER IMPLEMENTATION



LPF IMPLEMENTATION

BPF

FT IMPLEMENTATION

$$X[n,k] = \sum_{m=-\infty}^{\infty} w[n-m]\, x[m] e^{-j\omega_k m} e^{j\omega_k n} \cdot e^{-j\omega_k n}$$

$$= e^{-j\omega_k n} \sum_{m=-\infty}^{\infty} \underbrace{x[m]\, w[n-m]\, e^{+j\omega_k (n-m)}}_{x[n] * w[n] e^{j\omega_k n}}$$

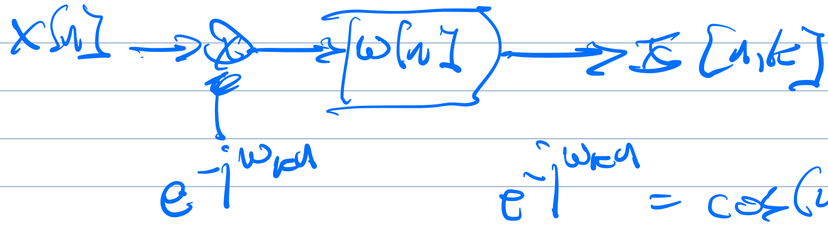
$$x[n] \longrightarrow \boxed{w[n] e^{j\omega_k n}} \longrightarrow \otimes \longrightarrow X[n,k]$$

$$e^{-j\omega_k n}$$

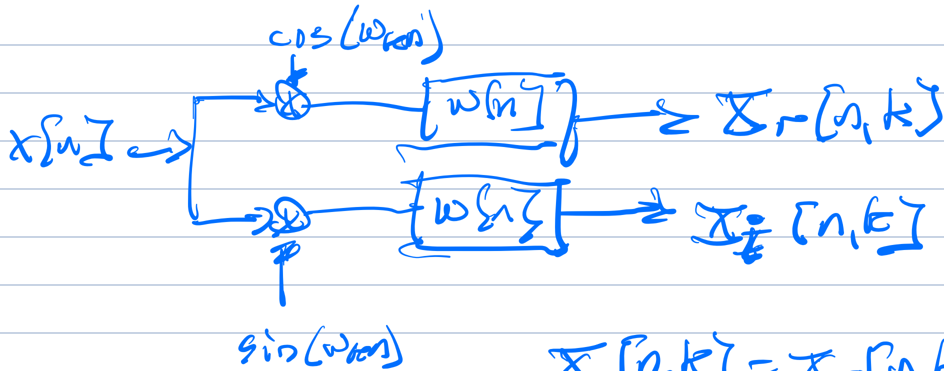
BANDPASS FILTER IMPLEMENTATION

IMPLEMENTING LPF, BPF VERSIONS OF STFT WITH REAL SIGNALS

LPF



$$e^{-j\omega_k n} = \cos(\omega_k n) - j\sin(\omega_k n)$$

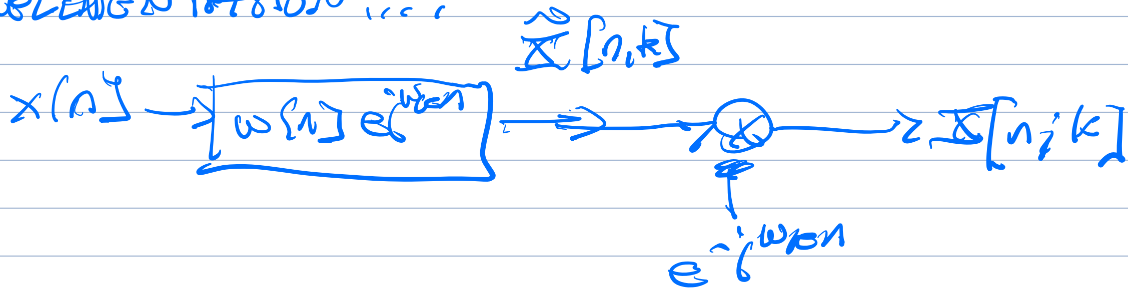


$$X[n,k] = X_r[n,k] - jX_i[n,k]$$

$$= x[n]\left(\cos(\omega_k n) - j\sin(\omega_k n)\right) * w[n]$$

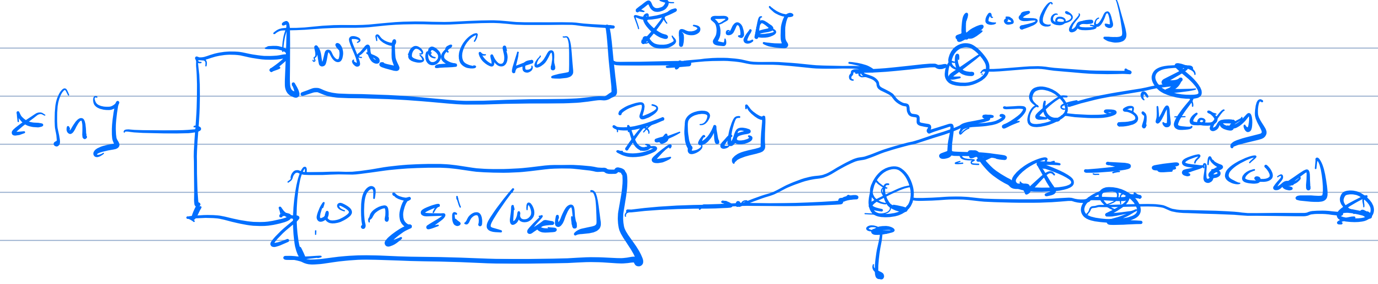
$$= \left(x[n]e^{-j\omega_k n}\right) * w[n] = X[n,k]$$

BPF IMPLEMENTATION ...



$$|\tilde{X}[n,k]| = |X[n,k]|$$

$$w[n]e^{j\omega_k n} = w[n]\cos(\omega_k n) + j\,w[n]\sin(\omega_k n)$$



$$\tilde{X}[n,k] = \tilde{X}_r[n,k] + j\,\tilde{X}_i[n,k]$$

SEE BETTER DRAWING IN NOTES ...