Laplace Mechanism

Let $f_{A(D)}(t)$ denote the density of the output of mechanism A applied to database D, evaluated at t. Recall mechanism A is defined as follows:

$$A(D) = f(D) + \mathrm{Lap}(\Delta/\epsilon)$$

where $f(D)$ is the query of interest, $\mathrm{Lap}(b)$ denotes 0-mean Laplacian noise of parameter b, and Δ is the global sensitivity of f. We want to show that $\forall$ neighboring databases D, D',

$$\frac{f_{A(D)}(t)}{f_{A(D')}(t)} \leq e^{\epsilon}.$$

Pf: Write the densities explicitly: $\dfrac{f_{A(D)}(t)}{f_{A(D')}(t)} = \dfrac{\exp\left\{-|f(D)-t| \Big/ \frac{\Delta}{\epsilon}\right\}}{\exp\left\{-|f(D')-t| \Big/ \frac{\Delta}{\epsilon}\right\}}$

$$= \exp\left\{\frac{\epsilon}{\Delta}\Big(|f(D')-t| - |f(D)-t|\Big)\right\} \leq \exp\left\{\frac{\epsilon}{\Delta}\,|f(D)-f(D')|\right\}$$

Use triangle inequality

$$\leq \exp\left\{\frac{\epsilon}{\Delta}\cdot\Delta\right\} = e^{\epsilon}. \qquad \blacksquare$$

Question: What if added noise has mean $\mu \neq 0$? $A'(D) = f(D) + \mathrm{Lap}(\mu, \Delta/\epsilon$

$$\frac{f_{A'(D)}(t)}{f_{A'(D')}(t)} = \frac{\exp\left\{-\frac{|f(D)+\mu-t|}{\Delta/\epsilon}\right\}}{\exp\left\{-\frac{|f(D')+\mu-t|}{\Delta/\epsilon}\right\}} \qquad \text{Let } q = t - \mu$$

Proof proceeds exactly as before. So this is also ϵ-DP.

Composition

Suppose we have k mechanisms that are $\epsilon_1, \dots, \epsilon_k$ DP, respectively:

$$A_1(D) = z_1 \qquad A_2(D, z_1) = z_2 \qquad \dots \qquad A_k(D, z_1, \dots, z_{k-1}) = z_k$$

$$f_{A_1, \dots, A_k, D}(\underline{t}) = f_{A_1(D)}(t_1) \cdot f_{A_2(D, t_1)}(t_2) \cdot \ \dots \ f_{A_k(D, t_1, \dots, t_{k-1})}(t_k)$$

$$\begin{array}{ccc} \leq & \leq & \leq \\ e^{\epsilon_1} \cdot f_{A_1(D')}(t_1) & e^{\epsilon_2} f_{A_2(D', t_1)}(t_2) & e^{\epsilon_k} \cdot f_{A_k(D', t_1, \dots, t_{k-1})}(t_k) \end{array}$$

$$= e^{\epsilon_1 + \dots + \epsilon_k} \cdot f_{A_1, \dots, A_k, D}(\underline{t}).$$

$\Rightarrow$ The composition of these mechanisms is $\sum_{i=1}^{k} \epsilon_i$ - DP.