18-600 Foundations of Computer Systems

Lecture 4: "Floating Point"

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September 12, 2016

- Required Reading Assignment:
  - **Chapter 2 of CS:APP (3rd edition) by Randy Bryant & Dave O'Hallaron**
- Assignments for This Week:
  - Lab 1

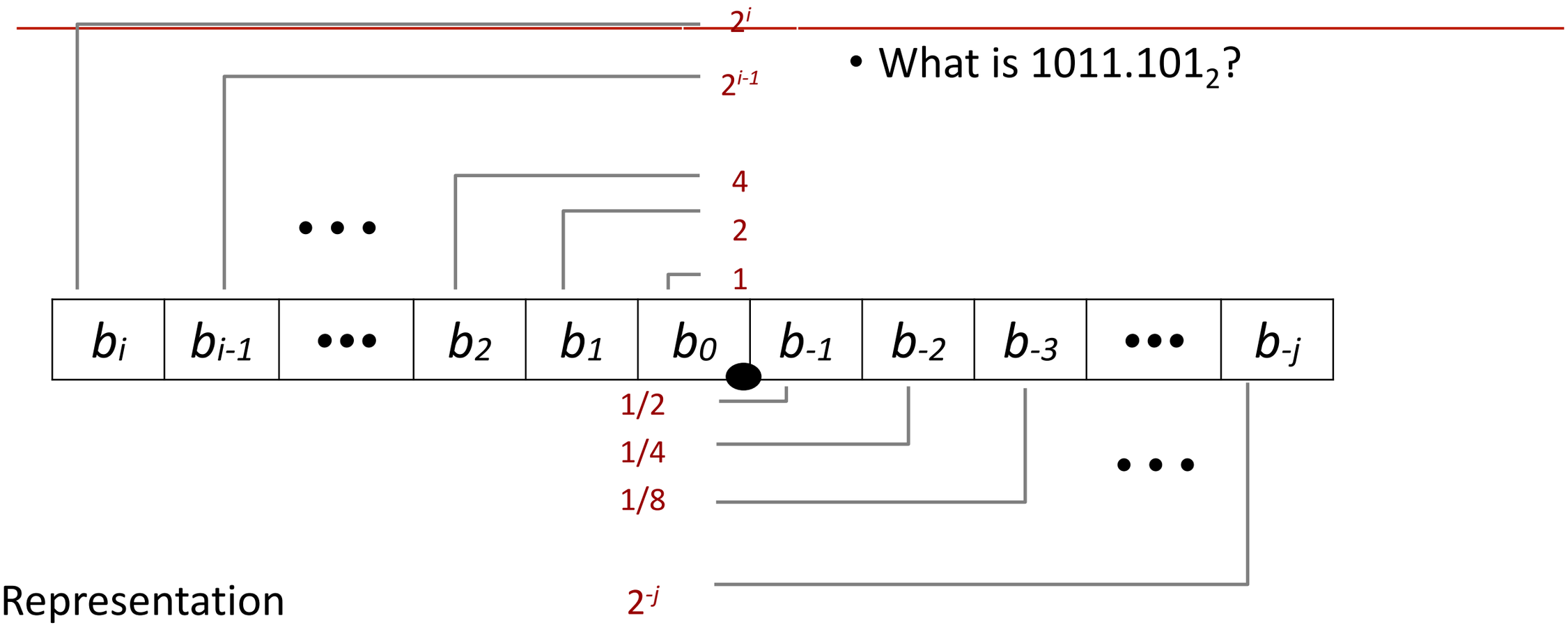
Electrical & Computer ENGINEERING

Carnegie Mellon University

Today: Floating Point

- Background: Fractional binary numbers
- IEEE floating point standard: Definition
- Example and properties
- Rounding, addition, multiplication
- Floating point in C
- Creating floating point number
- Summary of floating point number
- Quick introduction of assembly language

Fractional Binary Numbers



- What is 1011.101_2?

- Representation
  - Bits to right of "binary point" represent fractional powers of 2
  - Represents rational number:

$$\sum_{k=-j}^{i} b_k \times 2^k$$

Fractional Binary Numbers: Examples

Value	Representation
5 3/4	101.11_2
2 7/8	10.111_2
1 7/16	1.0111_2

- **Observations**
  - Divide by 2 by shifting right (unsigned)
  - Multiply by 2 by shifting left
  - Numbers of form $0.111111\ldots_2$ are just below 1.0
    - $1/2 + 1/4 + 1/8 + \ldots + 1/2^i + \ldots \rightarrow 1.0$
    - Use notation $1.0 - \varepsilon$

Representable Numbers

- Limitation #1
  - Can only exactly represent numbers of the form $x/2^k$
    - Other rational numbers have repeating bit representations
  - Value — Representation
    - 1/3 — `0.0101010101[01]`…$_2$
    - 1/5 — `0.001100110011[0011]`…$_2$
    - 1/10 — `0.0001100110011[0011]`…$_2$
- Limitation #2
  - Just one setting of binary point within the *w* bits
    - Limited range of numbers (very small values? very large?)

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IEEE Floating Point

- IEEE Standard 754
  - Established in 1985 as uniform standard for floating point arithmetic
    - Before that, many idiosyncratic formats
  - Supported by all major CPUs

- Driven by numerical concerns
  - Nice standards for rounding, overflow, underflow
  - Hard to make fast in hardware
    - Numerical analysts predominated over hardware designers in defining standard

Floating Point Representation

- Numerical Form:

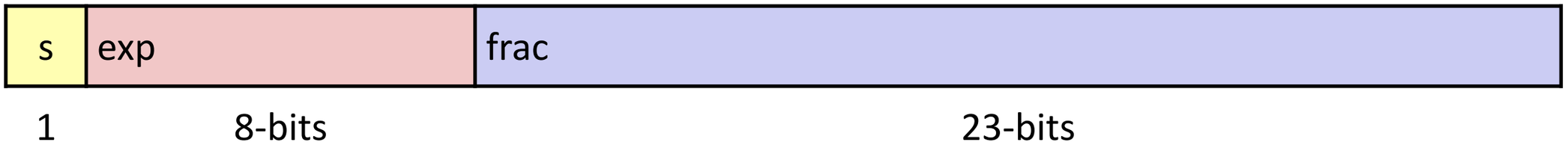
$$(-1)^s \, M \, 2^E$$

  - **Sign bit *s*** determines whether number is negative or positive
  - **Significand *M*** (mantissa) normally a fractional value in range [1.0,2.0).
  - **Exponent *E*** weights value by power of two

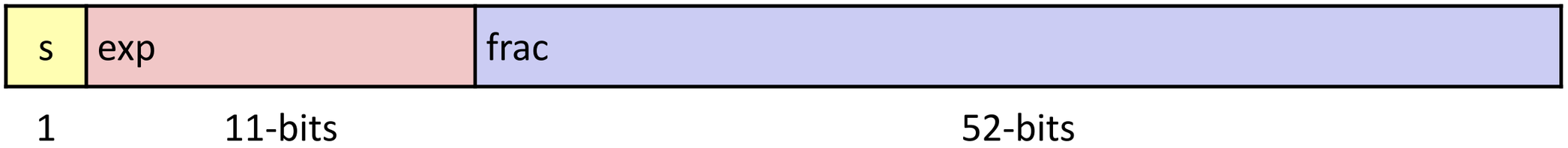
- Encoding
  - MSB s is sign bit ***s***
  - exp field encodes ***E*** (but is not equal to E)
  - frac field encodes ***M*** (but is not equal to M)

Precision options

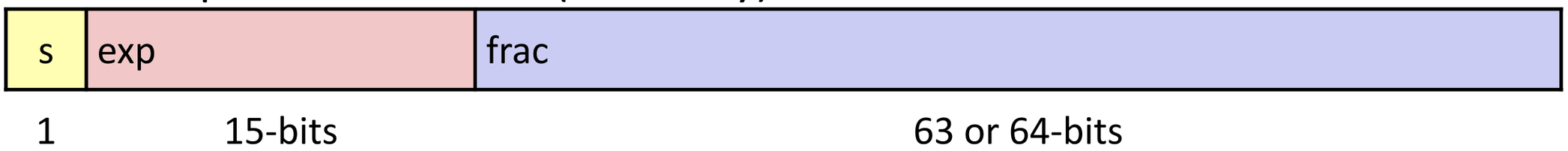
- Single precision: 32 bits



- Double precision: 64 bits



- Extended precision: 80 bits (Intel only)



"Normalized" Values

$$v = (-1)^s \, M \, 2^E$$

- When: exp ≠ 000…0 and exp ≠ 111…1

- Exponent coded as a ***biased*** value: ***E*** = ***Exp*** – ***Bias***
  - *Exp*: unsigned value of exp field
  - *Bias* = $2^{k-1} - 1$, where *k* is number of exponent bits
    - Single precision: 127 (Exp: 1…254, E: -126…127)
    - Double precision: 1023 (Exp: 1…2046, E: -1022…1023)

- Significand coded with implied leading 1: ***M*** = $1.xxx…x_2$
  - xxx…x: bits of frac field
  - Minimum when frac=000…0 (M = 1.0)
  - Maximum when frac=111…1 (M = 2.0 – ε)
  - Get extra leading bit for "free"

Normalized Encoding Example

$v = (-1)^s M 2^E$

$E = Exp - Bias$

- Value: `float F = 18600.0;`
  - $18600_{10} = 100100010101000_2 = 1.00100010101_2 \times 2^{14}$
- Significand

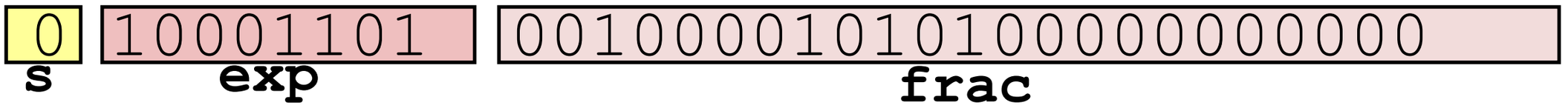
  $M = 1.\underline{00100010101}_2$

  frac $= \underline{00100010101}\mathbf{000000000000}_2$ **(23 bits)**
- Exponent

  $E = 14$

  $Bias = 127$

  $Exp = 141 =$ **10001101_2** **(8 bits)**
- Result:



Denormalized Values

$$v = (-1)^s M 2^E$$
$$E = 1 - Bias$$

- Condition: exp = 000…0

- Exponent value: ***E*** = 1 – Bias (instead of ***E*** = 0 – ***Bias***)
- Significand coded with implied leading 0: ***M*** = 0.xxx…x_2
  - **`xxx`**…**`x`**: bits of **`frac`**
- Cases
  - **`exp`** = **`000`**…**`0`**, **`frac`** = **`000`**…**`0`**
    - Represents zero value
    - Note distinct values: +0 and –0 (why?)
  - **`exp`** = **`000`**…**`0`**, **`frac`** ≠ **`000`**…**`0`**
    - Numbers closest to 0.0
    - Equispaced

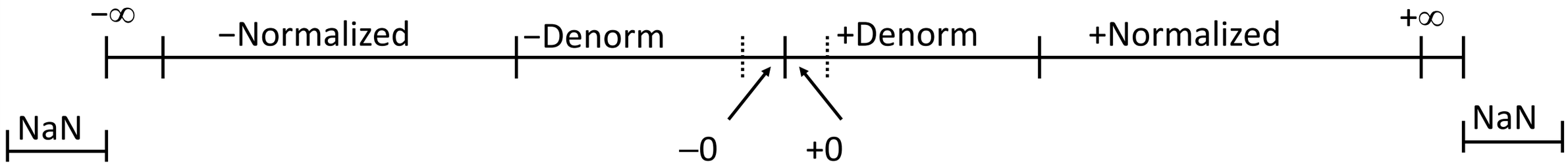
Special Values

- Condition: **`exp`** = **`111…1`**

- Case: **`exp`** = **`111…1`**, **`frac`** = **`000…0`**
  - Represents value ∞ (infinity)
  - Operation that overflows
  - Both positive and negative
  - E.g., 1.0/0.0 = −1.0/−0.0 = $+\infty$, 1.0/−0.0 = $-\infty$

- Case: **`exp`** = **`111…1`**, **`frac`** ≠ **`000…0`**
  - Not-a-Number (NaN)
  - Represents case when no numeric value can be determined
  - E.g., sqrt(–1), $\infty - \infty$, $\infty \times 0$

Visualization: Floating Point Encodings



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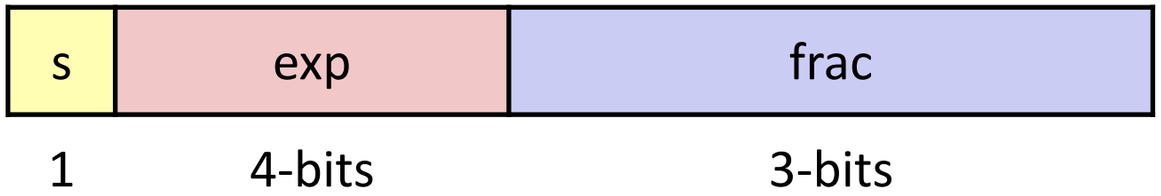
Tiny Floating Point Example



- 8-bit Floating Point Representation
  - the sign bit is in the most significant bit
  - the next four bits are the exponent, with a bias of 7
  - the last three bits are the **`frac`**

- Same general form as IEEE Format
  - normalized, denormalized
  - representation of 0, NaN, infinity

Dynamic Range (Positive Only)

$v = (-1)^s M 2^E$

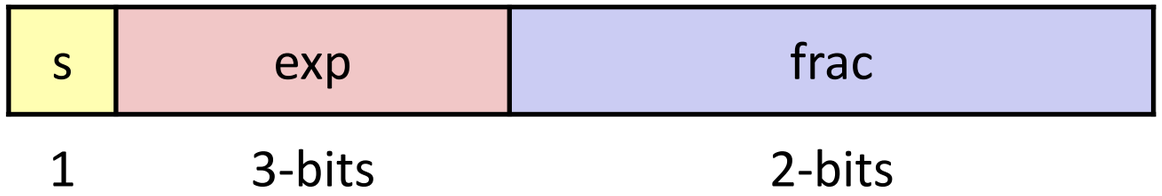
n: E = Exp – Bias

d: E = 1 – Bias

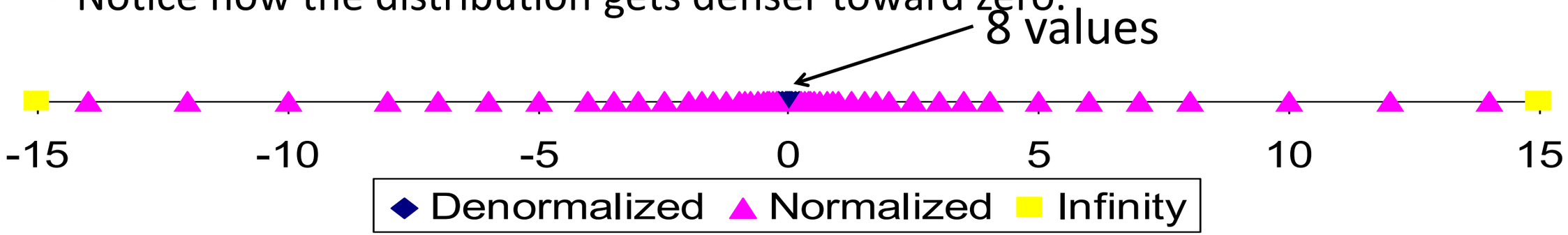
	s	exp	frac	E	Value	
	0	0000	000	-6	0	
	0	0000	001	-6	1/8*1/64 = 1/512	closest to zero
Denormalized numbers	0	0000	010	-6	2/8*1/64 = 2/512	
	…					
	0	0000	110	-6	6/8*1/64 = 6/512	
	0	0000	111	-6	7/8*1/64 = 7/512	largest denorm
	0	0001	000	-6	8/8*1/64 = 8/512	smallest norm
	0	0001	001	-6	9/8*1/64 = 9/512	
	…					
	0	0110	110	-1	14/8*1/2 = 14/16	
	0	0110	111	-1	15/8*1/2 = 15/16	closest to 1 below
Normalized numbers	0	0111	000	0	8/8*1 = 1	
	0	0111	001	0	9/8*1 = 9/8	closest to 1 above
	0	0111	010	0	10/8*1 = 10/8	
	…					
	0	1110	110	7	14/8*128 = 224	
	0	1110	111	7	15/8*128 = 240	largest norm
	0	1111	000	n/a	inf	

Distribution of Values

- 6-bit IEEE-like format
  - e = 3 exponent bits
  - f = 2 fraction bits
  - Bias is $2^{3-1}-1 = 3$

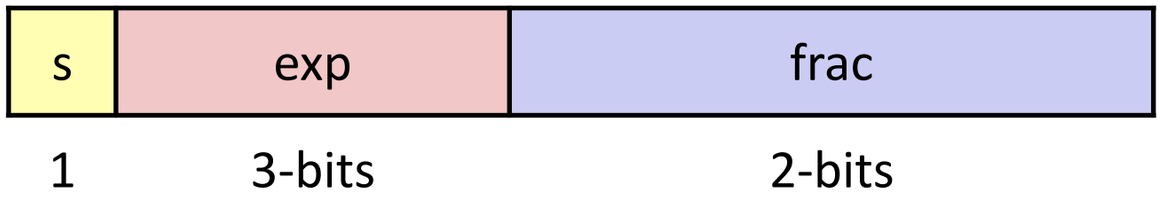


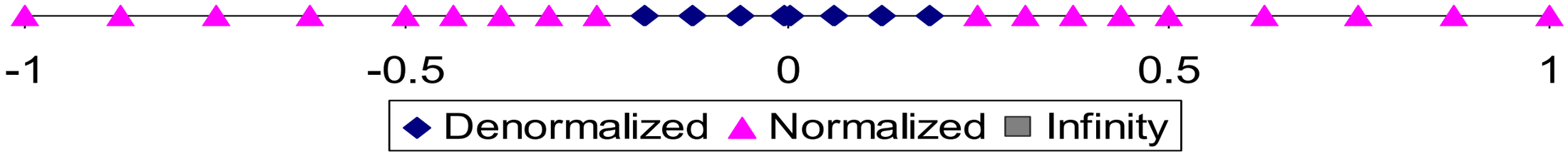
- Notice how the distribution gets denser toward zero.



Distribution of Values (close-up view)

- 6-bit IEEE-like format
  - e = 3 exponent bits
  - f = 2 fraction bits
  - Bias is 3





Special Properties of the IEEE Encoding

- FP Zero Same as Integer Zero
  - All bits = 0

- Can (Almost) Use Unsigned Integer Comparison
  - Must first compare sign bits
  - Must consider −0 = 0
  - NaNs problematic
    - Will be greater than any other values
    - What should comparison yield?
  - Otherwise OK
    - Denorm vs. normalized
    - Normalized vs. infinity

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Floating Point Operations: Basic Idea

- $x +_f y = \text{Round}(x + y)$
- $x \times_f y = \text{Round}(x \times y)$
- Basic idea
  - First compute exact result
  - Make it fit into desired precision
    - Possibly overflow if exponent too large
    - Possibly round to fit into **`frac`**

Rounding

- Rounding Modes (illustrate with $ rounding)

	$1.40	$1.60	$1.50	$2.50	–$1.50
• Towards zero	$1	$1	$1	$2	–$1
• Round down (−∞)	$1	$1	$1	$2	–$2
• Round up (+∞)	$2	$2	$2	$3	–$1
• Nearest Even (default)	$1	$2	$2	$2	–$2

Closer Look at Round-To-Even

- Default Rounding Mode
  - Hard to get any other kind without dropping into assembly
  - All others are statistically biased
    - Sum of set of positive numbers will consistently be over- or under- estimated

- Applying to Other Decimal Places / Bit Positions
  - When exactly halfway between two possible values
    - Round so that least significant digit is even
  - E.g., round to nearest hundredth

7.8949999	7.89	(Less than half way)
7.8950001	7.90	(Greater than half way)
7.8950000	7.90	(Half way—round up)
7.8850000	7.88	(Half way—round down)

Rounding Binary Numbers

- Binary Fractional Numbers
  - "Even" when least significant bit is **0**
  - "Half way" when bits to right of rounding position = $\mathbf{100}\ldots_2$

- Examples
  - Round to nearest 1/4 (2 bits right of binary point)

Value	Binary	Rounded	Action	Rounded Value
2 3/32	10.00011_2	10.00_2	(<1/2—down)	2
2 3/16	10.00110_2	10.01_2	(>1/2—up)	2 1/4
2 7/8	10.11100_2	11.00_2	(1/2—up)	3
2 5/8	10.10100_2	10.10_2	(1/2—down)	2 1/2

FP Multiplication

- $(-1)^{s1}$ ***M1*** 2^{E1} x $(-1)^{s2}$ ***M2*** 2^{E2}
- Exact Result: $(-1)^{s}$ ***M*** 2^{E}
  - Sign *s*: *s1* ^ *s2*
  - Significand *M*: *M1* x *M2*
  - Exponent *E*: *E1* + *E2*

- Fixing
  - If *M* ≥ 2, shift *M* right, increment *E*
  - If *E* out of range, overflow
  - Round *M* to fit **`frac`** precision

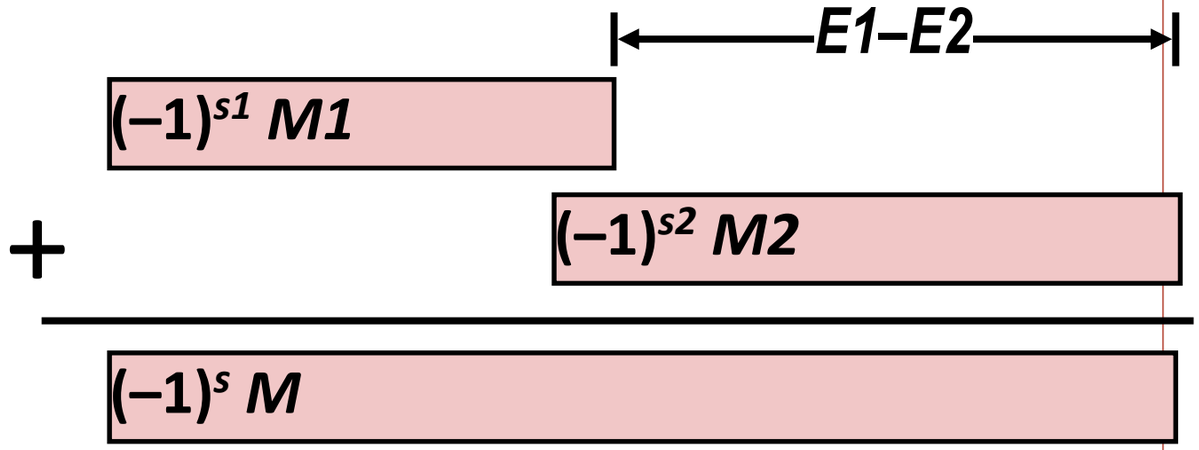
- Implementation
  - Biggest chore is multiplying significands

Floating Point Addition

- $(-1)^{s1}\ M1\ \ 2^{E1}\ \ +\ \ (-1)^{s2}\ M2\ \ 2^{E2}$
  - Assume *E1* > *E2*

- Exact Result: $(-1)^{s}\ M\ \ 2^{E}$
  - Sign *s*, significand *M*:
    - Result of signed align & add
  - Exponent *E*: *E1*

Get binary points lined up



- Fixing
  - If $M \geq 2$, shift *M* right, increment *E*
  - if $M < 1$, shift *M* left *k* positions, decrement *E* by *k*
  - Overflow if *E* out of range
  - Round *M* to fit **`frac`** precision

Mathematical Properties of FP Add

- Commutative? ***Yes***
- Associative? ***No***
  - Overflow and inexactness of rounding
  - `(3.14+1e10)-1e10 = 0, 3.14+(1e10-1e10) = 3.14`
- 0 is additive identity? ***Yes***
- Every element has additive inverse? ***Almost***
  - Yes, except for infinities & NaNs
- Monotonicity ***Almost***
  - $a \geq b \Rightarrow a+c \geq b+c$?
    - Except for infinities & NaNs

Mathematical Properties of FP Mult

- Multiplication Commutative? ***Yes***
- Multiplication is Associative? ***No***
  - Possibility of overflow, inexactness of rounding
  - Ex: `(1e20*1e20)*1e-20= inf, 1e20*(1e20*1e-20)= 1e20`
- 1 is multiplicative identity? ***Yes***
- Multiplication distributes over addition? ***No***
  - Possibility of overflow, inexactness of rounding
  - `1e20*(1e20-1e20)= 0.0, 1e20*1e20 – 1e20*1e20 = NaN`
- Monotonicity ***Almost***
  - $a \geq b$ & $c \geq 0 \Rightarrow a * c \geq b * c$?
    - Except for infinities & NaNs

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Floating Point in C

- C Guarantees Two Levels
  - **`float`** single precision
  - **`double`** double precision
- Conversions/Casting
  - Casting between **`int`**, **`float`**, and **`double`** changes bit representation
  - **`double`**/**`float`** → **`int`**
    - Truncates fractional part
    - Like rounding toward zero
    - Not defined when out of range or NaN: Generally sets to TMin
  - **`int`** → **`double`**
    - Exact conversion, as long as **`int`** has ≤ 53 bit word size
  - **`int`** → **`float`**
    - Will round according to rounding mode

Floating Point Puzzles

- For each of the following C expressions, either:
  - Argue that it is true for all argument values
  - Explain why not true

```
int x = …;
float f = …;
double d = …;
```

Assume neither
d nor **f** is NaN

- `x == (int)(float) x` ✗
- `x == (int)(double) x` ✓
- `f == (float)(double) f` ✓
- `d == (double)(float) d` ✗
- `f == -(-f);` ✓
- `2/3 == 2/3.0` ✗
- `d < 0.0 ⇒ ((d*2) < 0.0)` ✓
- `d > f ⇒ -f > -d` ✓
- `d * d >= 0.0` ✓
- `(d+f)-d == f` ✗

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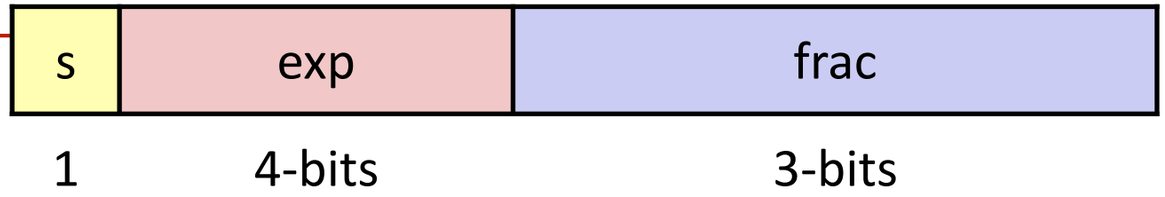
Creating Floating Point Number



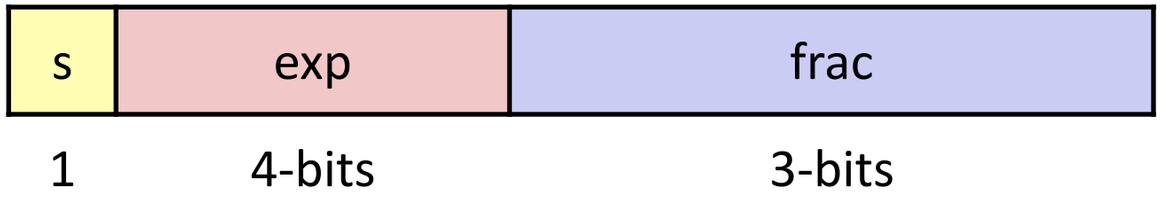
- Steps
  - Normalize to have leading 1
  - Round to fit within fraction
  - Postnormalize to deal with effects of rounding

- Case Study
  - Convert 8-bit unsigned numbers to tiny floating point format

  Example Numbers

128	10000000
15	00001101
17	00010001
19	00010011
138	10001010
63	00111111

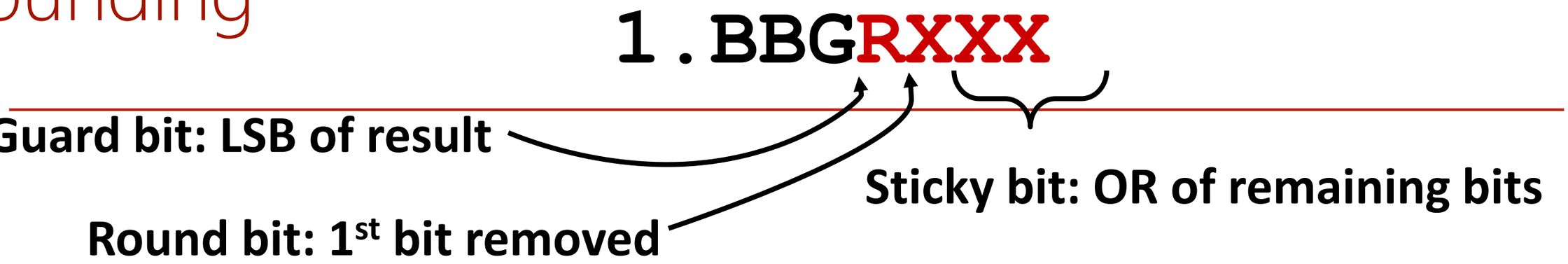
Normalize



- Requirement
  - Set binary point so that numbers of form 1.xxxxx
  - Adjust all to have leading one
    - Decrement exponent as shift left

Value	*Binary*	*Fraction*	*Exponent*
128	10000000	1.0000000	7
15	00001101	1.1010000	3
17	00010001	1.0001000	4
19	00010011	1.0011000	4
138	10001010	1.0001010	7
63	00111111	1.1111100	5

Rounding



- Round up conditions
  - Round = 1, Sticky = 1 → > 0.5
  - Guard = 1, Round = 1, Sticky = 0 → Round to even

Value	*Fraction*	*GRS*	*Incr?*	*Rounded*
128	1.0000000	000	N	1.000
15	1.1010000	100	N	1.101
17	1.0001000	010	N	1.000
19	1.0011000	110	Y	1.010
138	1.0001010	011	Y	1.001
63	1.1111100	111	Y	10.000

Postnormalize

- Issue
  - Rounding may have caused overflow
  - Handle by shifting right once & incrementing exponent

Value	*Rounded*	*Exp*	*Adjusted*	*Result*
128	1.000	7		128
15	1.101	3		15
17	1.000	4		16
19	1.010	4		20
138	1.001	7		134
63	10.000	5	1.000/6	64

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Summary

- IEEE Floating Point has clear mathematical properties
- Represents numbers of form M x 2^E
- One can reason about operations independent of implementation
  - As if computed with perfect precision and then rounded
- Not the same as real arithmetic
  - Violates associativity/distributivity
  - Makes life difficult for compilers & serious numerical applications programmers

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Compiling C Into Assembly

C Code (sum.c)

```
long plus(long x, long y);

void sumstore(long x, long y,
              long *dest)
{
    long t = plus(x, y);
    *dest = t;
}
```

Generated x86-64 Assembly

```
sumstore:
    pushq   %rbx
    movq    %rdx, %rbx
    call    plus
    movq    %rax, (%rbx)
    popq    %rbx
    ret
```

Obtain (on shark machine) with command

```
gcc -Og -S sum.c
```

Produces file `sum.s`

Machine Instruction Example

```
*dest = t;
```

- C Code
  - Store value **t** where designated by **dest**

```
movq %rax, (%rbx)
```

- Assembly
  - Move 8-byte value to memory
    - Quad words in x86-64 parlance
  - Operands:

    t: Register **%rax**

    dest: Register **%rbx**

    ***dest:** Memory **M[%rbx]**

```
0x40059e:  48 89 03
```

- Object Code
  - 3-byte instruction
  - Stored at address **0x40059e**

Assembly Characteristics: Operations

- Perform arithmetic function on register or memory data
- Transfer data between memory and register
  - Load data from memory into register
  - Store register data into memory
- Transfer control
  - Unconditional jumps to/from procedures
  - Conditional branches

Some Arithmetic Operations

- Two Operand Instructions:

Format	***Computation***	
`addq`	*Src,Dest* Dest = Dest + Src	
`subq`	*Src,Dest* Dest = Dest – Src	
`imulq`	*Src,Dest* Dest = Dest * Src	
`salq`	*Src,Dest* Dest = Dest << Src	***Also called shlq***
`sarq`	*Src,Dest* Dest = Dest >> Src	***Arithmetic***
`shrq`	*Src,Dest* Dest = Dest >> Src	***Logical***
`xorq`	*Src,Dest* Dest = Dest ^ Src	
`andq`	*Src,Dest* Dest = Dest & Src	
`orq`	*Src,Dest* Dest = Dest | Src	

- One Operand Instructions

`incq`	*Dest*	*Dest = Dest + 1*
`decq`	*Dest*	*Dest = Dest – 1*
`negq`	*Dest*	*Dest = – Dest*
`notq`	*Dest*	*Dest = ~Dest*

x86-64 Integer Registers

%rax	%eax	%r8	%r8d
%rbx	%ebx	%r9	%r9d
%rcx	%ecx	%r10	%r10d
%rdx	%edx	%r11	%r11d
%rsi	%esi	%r12	%r12d
%rdi	%edi	%r13	%r13d
%rsp	%esp	%r14	%r14d
%rbp	%ebp	%r15	%r15d

Simple Memory Addressing Modes

- Normal (R) Mem[Reg[R]]
  - Register R specifies memory address
  - Pointer dereferencing in C

```
movq (%rcx),%rax
```

- Displacement D(R) Mem[Reg[R]+D]
  - Register R specifies start of memory region
  - Constant displacement D specifies offset

```
movq 8(%rbp),%rdx
```

18-600 Foundations of Computer Systems

Lecture 5:
"Machine Programs I: (Basics)"

John Shen & Zhiyi Yu
September 14, 2016

Next Time ...

Electrical & Computer ENGINEERING