3/15/24

RECITATION 6B

LINEAR V. CIRCULAR CONVOLUTION, OLA, OLS

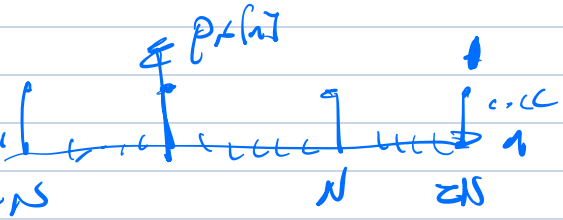
FFT ALGORITHMS

LIN V. CIRC. CONV.

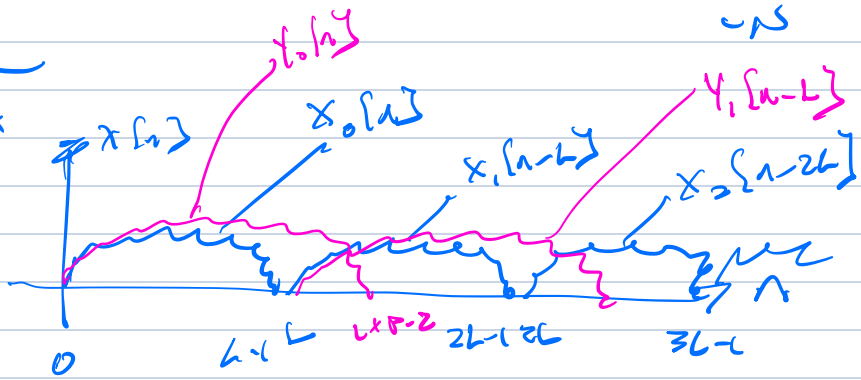
$x_1[n] * x_2[n] = y[n]$

$N_1 \quad N_2 \quad N_1+N_2-1$

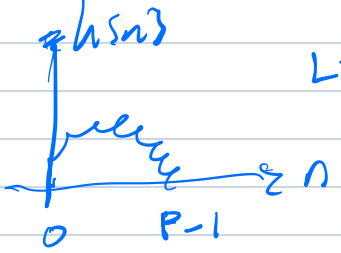
N-PT. CIRC. CONV

$x_1[n] \overset{N}{\circledast} x_2[n] = x_1[n] * x_2[n] +$ 

OLA



let $x_i[n] = \begin{cases} x[n+iL], & 0 \le n \le L-1 \\ 0, & \text{ELSE} \end{cases}$

ASSUME 

$$x[n] = \sum_{i=0}^{\infty} x_i[n-iL]$$

OLA DFT SIZE $N = L+P-1$

$$y_i[n] = x_i[n] * h[n] \qquad 0 \le n \le L+P-2$$

$$y[n] = \sum_{i=0}^{\infty} y_i[n - iL]$$

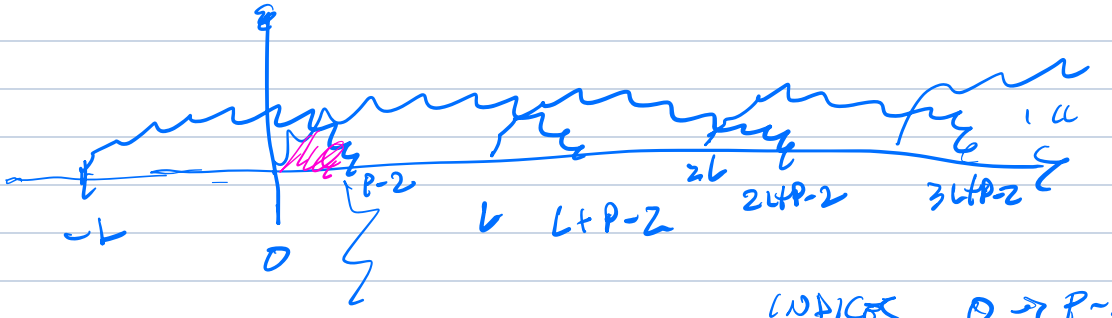
OLA Ex. $N = 512$, $P = 100$

$L+P-1 = N$; $L = N-(P-1) = 512-99 = 413$

	INPUT		OUTPUT	
i	BEG	END	BEG	END
0	0	412	0	511
1	413	825	413	924
2	826	1238	826	1337

OLS INPUT SEQ. $h[n]$ LENGTH L
LENGTH P

DFT SIZE L



$-L+L+P-2 = P-2$

INDICES $0 \to P-2$ DISTORTED
$P-1 \to L-1$ OK

OLS $L = 512$, $P = 100$

	INPUT		OUTPUT SAVED	
i	BEGIN	END	BEG	END
0	0	511	99	511
1	413	924	512	924
2	826	1337	925	1337

COMPUTATIONAL COST

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{nk}$$

$x_1[n] * x_2[n]$ COST N^2

N N

N-PT. DFT COST N^2

CONVOLUTION USING N-PT. DFTs COST $3N^2 + N$

CONVOLUTION USING N-PT. FFTs. COST $3\left(\frac{N}{2}\log_2(N)\right) + N$

$N^2 \to \frac{N}{2}\log_2 N$

OLS/OLA USING # SEGMENTS

$$3\left(\frac{N}{2}\log_2(N) + N\right) + \frac{N}{2}\log_2(N)$$

COOLEY-TUKEY FFT

$$X[k] = \sum_{n=0}^{N-1} x(n) W_N^{nk}$$

$$= \sum_{\substack{n=0 \\ n \text{ EVEN}}}^{N-1} x[n] W_N^{nk} + \sum_{\substack{n=0 \\ n \text{ ODD}}}^{N-1} x[n] W_N^{nk}$$

$n = 2r$ $\qquad$ $n = 2r+1$

$$= \sum_{r=0}^{\frac{N}{2}-1} x[2r] W_N^{2rk} + \sum_{r=0}^{\frac{N}{2}-1} x[2r+1] W_N^{(2r+1)k}$$

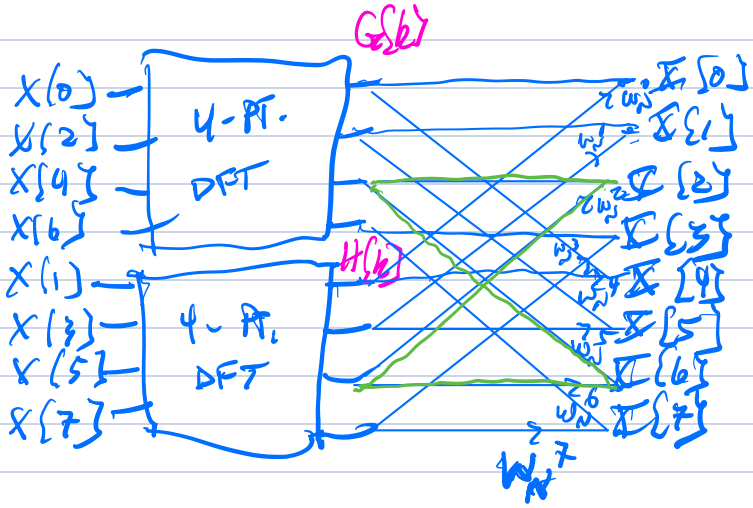
$$W_N^{2rk} = e^{-j\frac{2\pi}{N}2rk} = e^{-j\frac{2\pi}{N/2}rk} = W_{\frac{N}{2}}^{rk}$$

$$W_N^{(2r+1)k} = W_N^{2rk} W_N^{k} = W_{\frac{N}{2}}^{rk} W_N^{k}$$

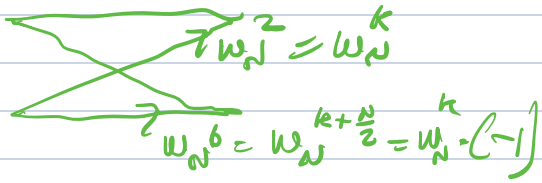
$$X[k] = \sum_{r=0}^{\frac{N}{2}-1} x[2r] W_{\frac{N}{2}}^{rk} + W_N^{k} \sum_{r=0}^{\frac{N}{2}-1} x[2r+1] W_{\frac{N}{2}}^{rk}$$

$G[k]$ $\frac{N}{2}$-PT. DFT of $x[2r]$

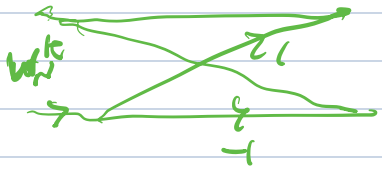
$H[k]$ $\frac{N}{2}$-PT. DFT of $x[2r+1]$



$$X[k] = G[k] + W_N^k H[k]$$



NON-RADIX-2 FFTs ...



$N = p_1 p_2 p_3 \cdots p_\nu$

$q_1 = \frac{N}{p_1}$, $q_2 = \frac{N}{p_1 p_2}$ ETC.

FOR $N = p_1 q_1$

$$X[k] = \sum_{r=0}^{q_1-1} x[p_1 r] W_N^{p_1 rk} + \sum_{r=0}^{q_1-1} x[p_1 r+1] W_N^{(p_1 r+1)k} + \cdots + \sum_{r=0}^{q_1-1} x[p_1 r+(p_1-1)] W_N^{(p_1 r+(p_1-1))k}$$

$$\sum_{r=0}^{q_1-1} x[p_1 r+\ell] W_N^{(p_1 r+\ell)k}$$

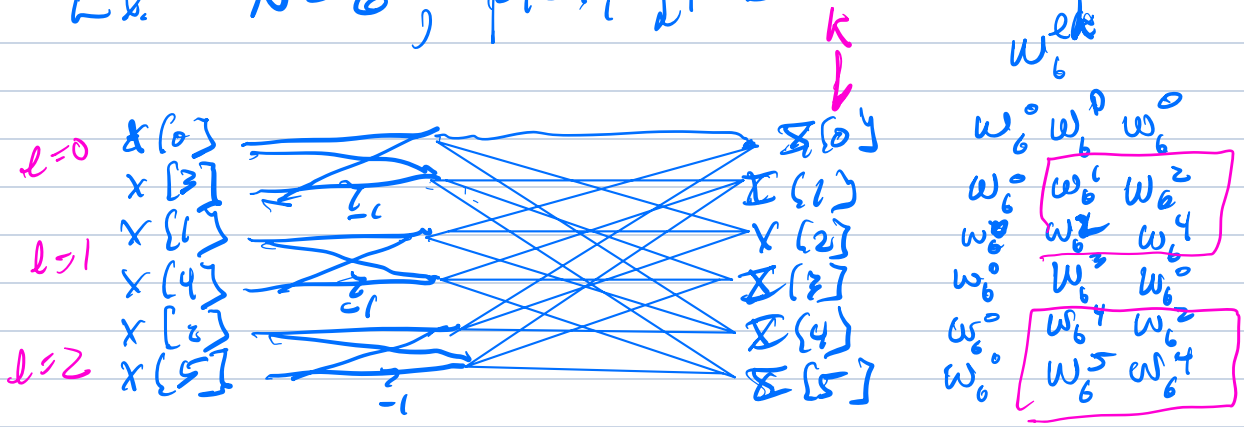
$$W_N^{(p_1 r+\ell)k} = W_N^{p_1 rk}\, W_N^{\ell k} = W_{\frac{N}{p_1}}^{rk}\, W_N^{\ell k}$$

$$= W_{q_1}^{rk}\, W_N^{\ell k}$$

$$X[k] = \sum_{\ell=0}^{p_1-1} W_N^{\ell k} \cdot \sum_{r=0}^{q_1-1} x[p_1 r+\ell]\, W_{q_1}^{rk}$$

q_1-PT. DFT of $x[p_1 r+\ell]$

Ex. $N=6$, $p_1=3$, $q_1=2$



TOTAL $W_N^{\ell k} = 12$

8 NONTRIVIAL MULTS EXCLUDING ± 1