2/9/24 RECITATION 3B

ALLPASS, MINIMUM PHASE, LINEAR PHASE SYSTEMS;

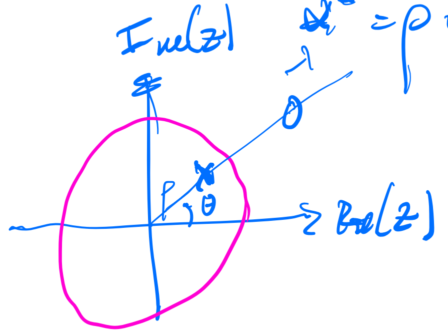
Z-TRANSFORMS + INVERSES

ALLPASS SYSTEMS

CONSIDER
$$H(z) = \frac{z^{-1} - \alpha^*}{1 - \alpha z^{-1}} = \frac{1 - z\alpha^*}{z - \alpha}$$

Let $\alpha = \rho e^{j\theta}$

$\alpha^* = \rho e^{-j\theta}$



POLE @ $z = \alpha = \rho e^{j\theta}$

ZERO @ $1 - z\alpha^* = 0$

$z\alpha^* = 1$; $z = \frac{1}{\alpha^*} = \frac{1}{\rho e^{-j\theta}} = \frac{1}{\rho} e^{j\theta}$

POLE @ $\rho e^{j\theta}$

ZERO @ $\frac{1}{\rho} e^{j\theta}$

$\Big\}$ "MIRROR IMAGE LOCATIONS"

$$H(e^{j\omega}) = H(z)\Big|_{z = e^{j\omega}}$$

$$|H(e^{j\omega})|^2 = H(e^{j\omega}) \cdot H^*(e^{j\omega}) = \left(\frac{e^{-j\omega} - \rho e^{-j\theta}}{1 - \rho e^{j\theta} e^{-j\omega}}\right) \cdot \left(\frac{e^{j\omega} - \rho e^{j\theta}}{1 - \rho e^{-j\theta} e^{j\omega}}\right)$$

$$= \frac{1 - \rho e^{j\theta} e^{-j\omega} - \rho e^{-j\theta} e^{j\omega} + \rho^2}{1 - \rho e^{-j\theta} e^{j\omega} - \rho e^{j\theta} e^{-j\omega} + \rho^2} = 1$$

ALLPASS FILTER

$$|H(e^{j\omega})| = 1$$

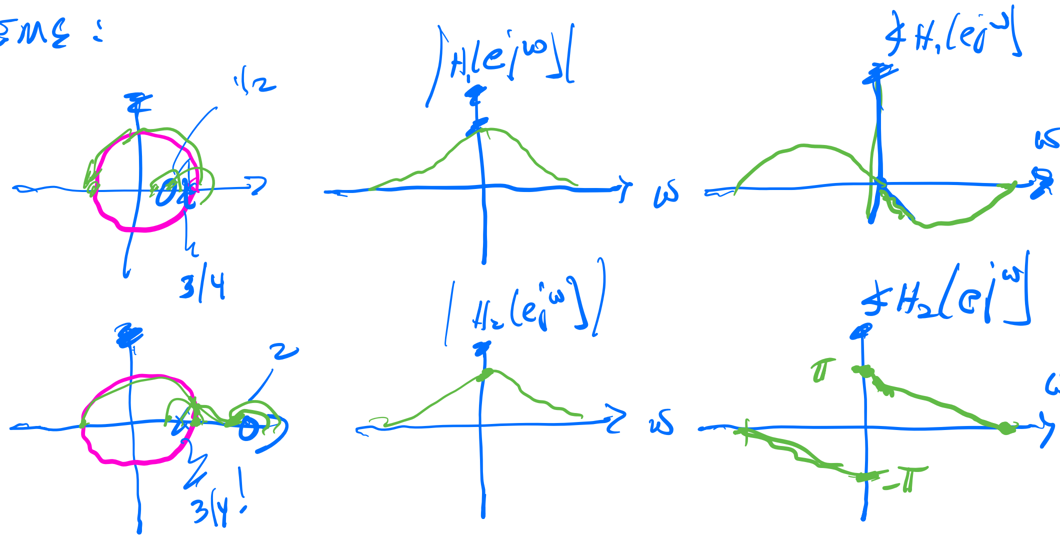
IF ALL POLES + ZEROS OF AN LSI SYSTEM OCCUR IN MIRROR-IMAGE LOCATIONS

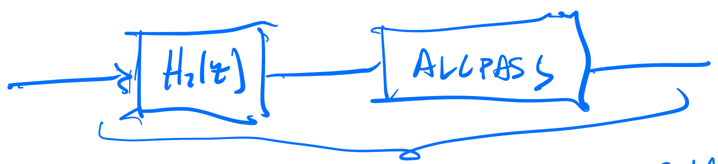
THE SYSTEM IS ALL PASS

CONSIDER TWO SYSTEMS:

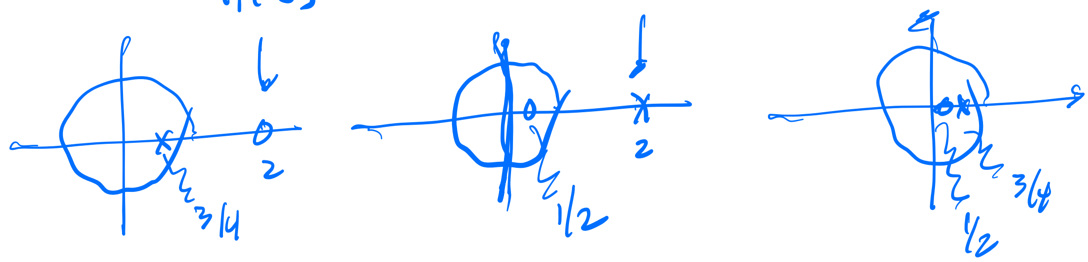
1. $H_1(z) = \dfrac{z-\frac{1}{2}}{z-\frac{3}{4}}$

2. $H_2(z) = \dfrac{z-2}{z-\frac{3}{4}}$





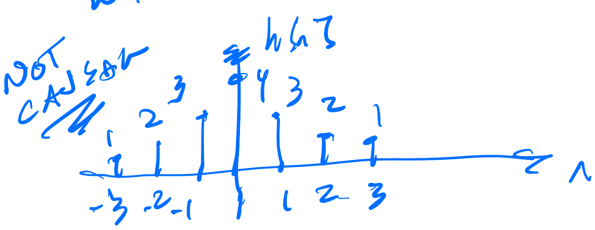
$H_1(z)$ MINIMUM PHASE SYSTEM



$H_2(z)$

MIN-PHASE SYSTEM: ALL ZEROS + ALL POLES INSIDE THE U.C.

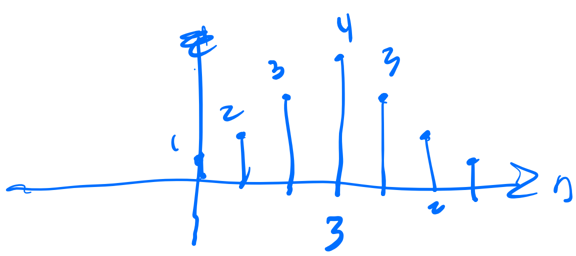
LINEAR PHASE SYSTEMS



$\Longleftrightarrow$ $H(e^{j\omega})$ REAL EVEN

ZERO PHASE

Let $h'[n] = h[n-3]$



$\Longleftrightarrow$ $H'(e^{j\omega}) = H(e^{j\omega})\, e^{-j\omega 3}$

$\measuredangle H'(e^{j\omega}) = \measuredangle H(e^{j\omega}) - 3\omega$

$$H'(z) = 1+2z^{-1}+3z^{-2}+4z^{-3}+3z^{-4}+2z^{-5}+z^{-6}$$

ZEROS ARE SOLUTIONS TO

$$1+2z^{-1}+3z^{-2}+4z^{-3}+3z^{-4}+2z^{-5}+z^{-6} = 0$$

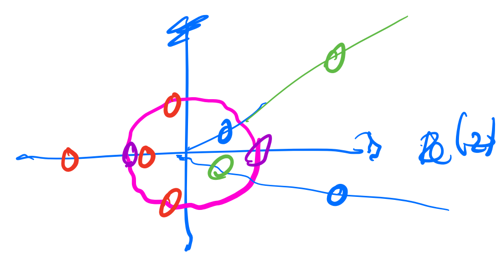
MULT BY z^6

$$z^6+2z^5+3z^4+4z^3+3z^2+2z+1 = 0$$

IF z_0 IS A ZERO OF $H'(z)$, $\frac{1}{z_0}$ IS A ZERO AS WELL!

CONSIDER $z_0 = \rho e^{j\theta}$

$$\frac{1}{z_0} = \frac{1}{\rho e^{j\theta}} = \frac{1}{\rho} e^{j\theta}$$



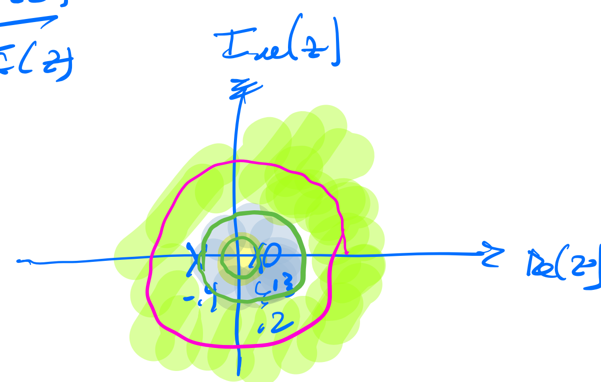
"CONJUGATE RECIPROCAL PAIRS"

INVERSE Z-TRANSFORMS

$$x[n] \rightarrow \boxed{h[n]} \rightarrow y[n]$$

$$Y(z) = X(z) \cdot H(z), \quad H(z) = \frac{Y(z)}{X(z)}$$

CONSIDER $H(z) = \dfrac{1-.3z^{-1}}{(1-.2z^{-1})(1+.4z^{-1})}$



POSSIBLE ROCs

I. $|z| > .4$ $h[n]$ RIGHT-SIDED (POTENTIALLY CAUSAL) STABLE

II. $|z| < .2$ $h[n]$ LEFT-SIDED, NOT CAUSAL, NOT STABLE

III. $.2 < |z| < .4$ $h[n]$ BOTH-SIDED, NOT CAUSAL NOT STABLE

IZTs USING PARTIAL FRACTIONS

$$H(z) = \frac{1 - .3z^{-1}}{(1 - .2z^{-1})(1 + .4z^{-1})} \qquad M = 1,\ N = 2$$

$$H(z) = \frac{A_1}{1 - .2z^{-1}} + \frac{A_2}{1 + .4z^{-1}} = \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}$$

↑ POLES

"RESIDUES"

$$A_k = H(z)(1 - d_k z^{-1})\Big|_{z = d_k}$$

$$A_1 = \frac{(1 - .3z^{-1})\cancel{(1 - .2z^{-1})}}{\cancel{(1 - .2z^{-1})}(1 + .4z^{-1})}\Bigg|_{z = d_k = .2} = \frac{1 - \frac{.3}{.2}}{1 + \frac{.4}{.2}} = \frac{-1/2}{3} = -\frac{1}{6}$$

$$A_2 = \frac{(1 - .3z^{-1})\cancel{(1 + .4z^{-1})}}{(1 - .2z^{-1})\cancel{(1 + .4z^{-1})}}\Bigg|_{z = -.4} = \frac{1 + \frac{.3}{.4}}{1 + \frac{.2}{.4}} = \frac{7/4}{6/4} = \frac{7}{6}$$

$$H(z) = -\frac{1}{6}\,\frac{1}{1 - .2z^{-1}} + \frac{7}{6}\,\frac{1}{1 + .4z^{-1}}$$

$$\alpha^n u[n] \Leftrightarrow \frac{1}{1 - \alpha z^{-1}}, \quad |z| > |\alpha|$$

$$-\alpha^n u[-n-1] \Leftrightarrow \frac{1}{1 - \alpha z^{-1}}, \quad |z| < |\alpha|$$

I. $|z| > .4 \Rightarrow h[n] = -\frac{1}{6}(.2)^n u[n] + \frac{7}{6}(-.4)^n u[n]$

II. $|z| < .2 \Rightarrow h[n] = \frac{1}{6}(.2)^n u[-n-1] - \frac{7}{6}(-.4)^n u[-n-1]$

III. $.2 < |z| < .4 \Rightarrow h[n] = -\frac{1}{6}(.2)^n u[n] - \frac{7}{6}(-.4)^n u[-n-1]$

ZTs + DIFF. EQS

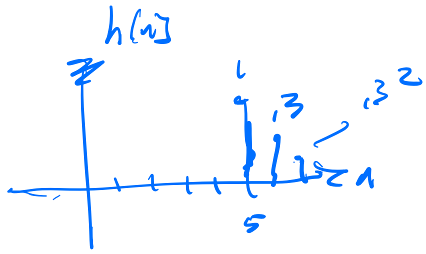
$$H(z) = \frac{1-.3z^{-1}}{(1-.2z^{-1})(1+.4z^{-1})} = \frac{1-.3z^{-1}}{1+.2z^{-1}-.08z^{-2}} = \frac{Y(z)}{X(z)}$$

$$Y(z)(1+.2z^{-1}-.08z^{-2}) = X(z)(1-.3z^{-1})$$

$$y[n]+.2y[n-1]-.08y[n-2] = x[n]-.3x[n-1]$$

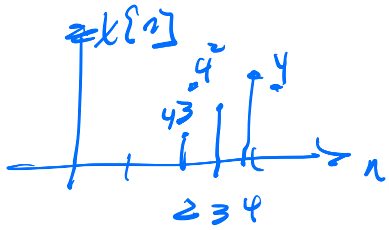
$$y[n] = -.2y[n-1]+.08y[n-2]+x[n]-.3x[n-1]$$

$$x[n] \rightarrow \boxed{h[n]} \rightarrow y[n]$$



Let $h[n] = (.3)^{n-5}u[n-5]$

$$x[n] = (.4)^{-n+5}u[-n+4]$$

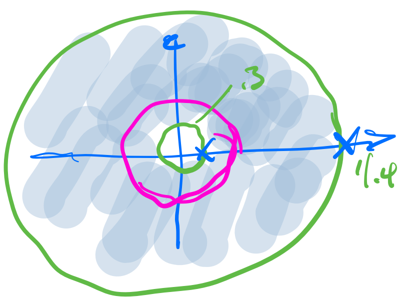


$$H(z) = \frac{z^{-5}}{1-.3z^{-1}}, \quad \text{ROC } |z| > .3$$

$$x[n] = (.4)^{-n+5}u[-n+4] = \left(\frac{1}{.4}\right)^{n-5}u[-n+4]$$

$$X(z) = z^{-5}\frac{1}{1-\frac{1}{.4}z^{-1}}, \quad |z| < \frac{1}{.4}$$

$$Y(z) = H(z)X(z) = z^{-10}\frac{1}{1-.3z^{-1}} \cdot \frac{1}{1-\frac{1}{.4}z^{-1}}$$



$$y[n] = \underbrace{\left\{A_1 (.3)^n u[n] + A_2\left(-\left(\frac{1}{.4}\right)^n\right)u[-n-1]\right\}}_{A}$$

SHIFTED TO THE RIGHT BY 10 SAMPLES

$$X(z) = \sum_{n=-\infty}^{\infty}\left(\frac{1}{.4}\right)^{n-5} z^{-n} u[-n+4] = \sum_{n=4}^{-\infty}\left(\frac{1}{.4}\right)^{n-5} z^{-n}$$

$$= (.4)^5 \cdot \sum_{l=-4}^{\infty} \left(\frac{1}{.4}\right)^{-l} z^{+l}$$

$l = -n$

$n = -l$

$$= (.4)^5 \sum_{r=0}^{\infty} \left(\frac{1}{.4}\right)^{-(r-4)} z^{r-4} = .4^5 \left(\frac{1}{.4}\right)^4 z^{-4} \sum_{r=0}^{\infty} (.4z)^r$$

$r = l+4$

$l = r-4$

$$= .4z^{-4} \frac{1}{1-.4z}$$

$$\frac{z^{-5}}{1-\frac{1}{.4}z^{-1}}$$

$$= \frac{z^{-5}}{\frac{1}{.4z^{-1}}(1-.4z)} = \frac{z^{-5}}{\frac{1}{.4z^{-1}}-1}$$