2/9/24 RECITATION 3A

ALLPASS, MIN PHASE, LIN PHASE SYSTEMS;
PROBLEM SET 3: Z TRANSFORMS & INVERSES

ALLPASS SYSTEMS

CONSIDER

$$H(z) = \frac{z^{-1} - \alpha^*}{1 - \alpha z^{-1}} = \frac{1 - z\alpha^*}{z - \alpha}$$

Let $\alpha = \rho e^{j\theta}$

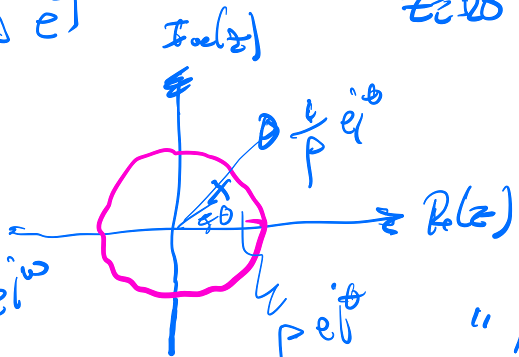
$\alpha^* = \rho e^{-j\theta}$

POLE AT $z = \alpha$

ZERO AT $1 - z\alpha^* = 0$

$z\alpha^* = 1$; $z = \frac{1}{\alpha^*} = \frac{1}{\rho e^{-j\theta}}$

$= \frac{1}{\rho} e^{j\theta}$



$H(e^{j\omega}) = H(z)\big|_{z = e^{j\omega}}$

"MIRROR-IMAGE" POLE-ZERO LOCATIONS

$$|H(e^{j\omega})|^2 = H(e^{j\omega}) \cdot H^*(e^{j\omega})$$

$$= \frac{1 - e^{j\omega}\rho e^{-j\theta}}{e^{j\omega} - \rho e^{j\theta}} \cdot \frac{1 - e^{-j\omega}\rho e^{j\theta}}{e^{-j\omega} - \rho e^{-j\theta}}$$

$$= \frac{1 - e^{-j\omega}\rho e^{j\theta} - e^{j\omega}\rho e^{-j\theta} + \rho^2}{1 - \rho e^{-j\theta} e^{j\omega} - \rho e^{j\theta} e^{-j\omega} + \rho^2} = 1!$$

ALLPASS SYSTEM $\Rightarrow |H(e^{j\omega})| = 1$

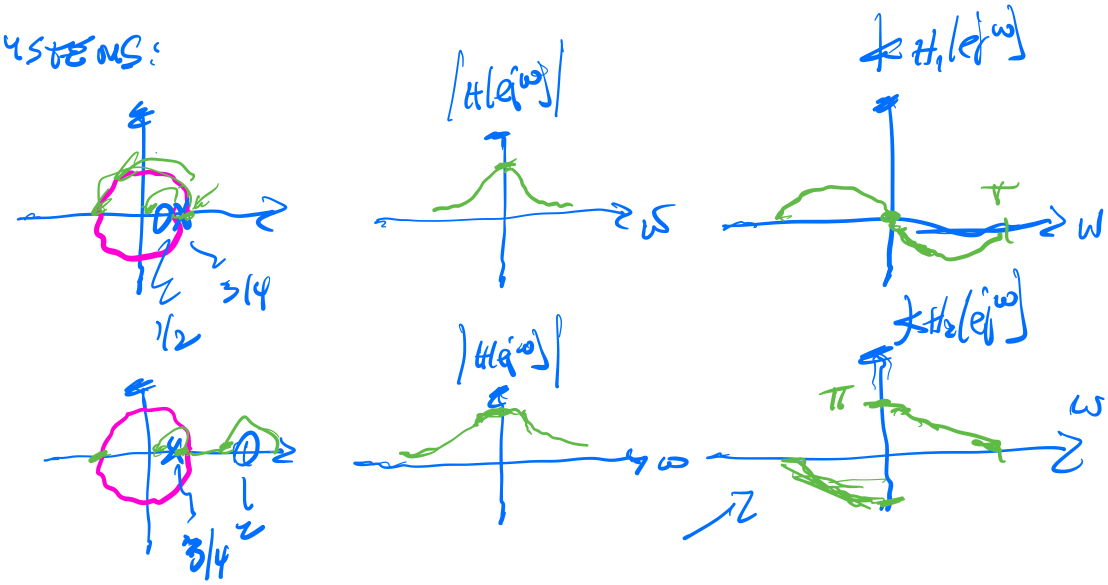
IF POLES, ZEROS ARE MIRROR-IMAGE PAIRS, $|H(e^{j\omega})| = \text{CONSTANT}$

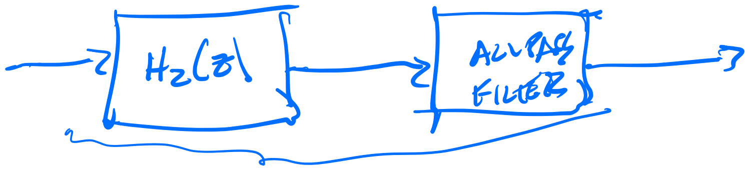
CONSIDER TWO SYSTEMS:

1. $H_1(z) = \dfrac{z - \frac{1}{2}}{z - \frac{3}{4}}$

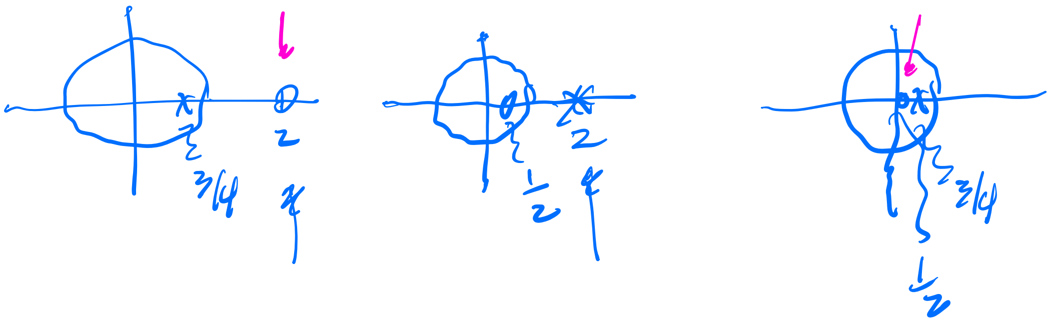
"MINIMUM PHASE"
(ALL ZEROS INSIDE U.C.)

2. $H_2(z) = \dfrac{z - 2}{z - \frac{3}{4}}$



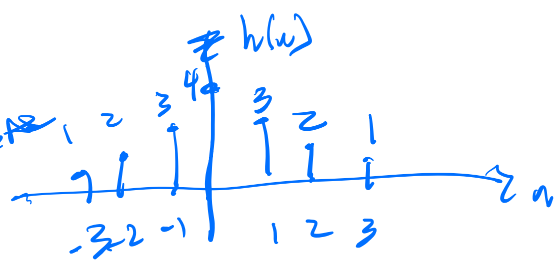


MINIMUM PHASE SYSTEM



LINEAR-PHASE SYSTEMS

BUT NOT CAUSAL

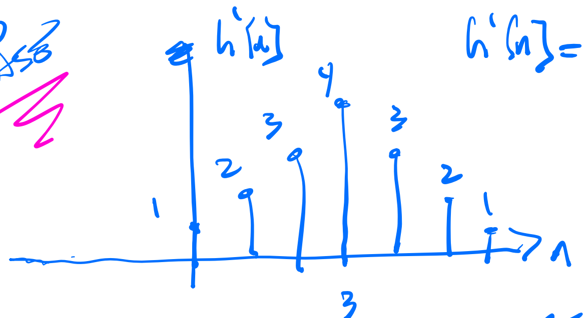


$h[n]$ REAL, EVEN

$\Longleftrightarrow H(e^{j\omega})$ REAL, EVEN

$H(e^{j\omega})$ "ZERO PHASE"

ZERO PHASE



$h'[n] = h[n-3]$

$H'(e^{j\omega}) = H(e^{j\omega})\, e^{-j3\omega}$

$\angle H'(e^{j\omega}) = \angle H(e^{j\omega}) - 3\omega$

LINEAR-PHASE SYSTEM

POLE LOCATIONS OF LINEAR-PHASE SYSTEMS

CONSIDER $H'(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 3z^{-4} + 2z^{-5} + z^{-6}$

ZEROS of $H'(z)$

SOLUTION TO $1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 3z^{-4} + 2z^{-5} + z^{-6} = 0$

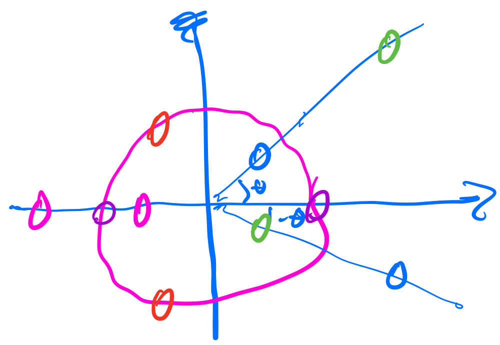
MULT BY z^6 $z^6 + 2z^5 + 3z^4 + 4z^3 + 3z^2 + 2z + 1 = 0$

IF $z = z_0$ IS A POLE OR ZERO

THEN $z = \frac{1}{z_0}$ IS ALSO A POLE OR ZERO

CONSIDER $z_0 = \rho e^{j\theta}$

$$\frac{1}{z_0} = \frac{1}{\rho e^{j\theta}} = \frac{1}{\rho} e^{-j\theta}$$



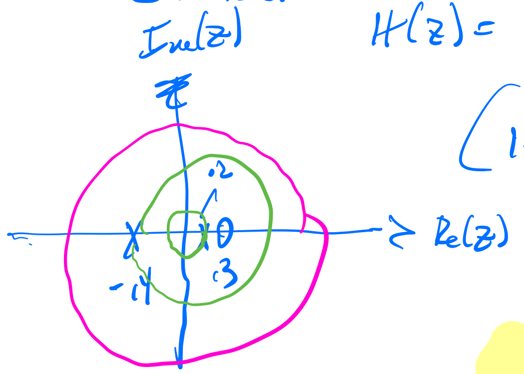
"CONJUGATE RECIPROCAL LOCATIONS"

INVERSE Z-TRANSFORMS

$$x[n] \rightarrow \boxed{h[n]} \rightarrow y[n]$$

$$Y(z) = X(z) \cdot H(z), \quad H(z) = \frac{Y(z)}{X(z)}$$

CONSIDER



$$H(z) = \frac{1-.3z^{-1}}{(1-.2z^{-1})(1+.4z^{-1})} = \frac{z(z-.3)}{(z-.2)(z+.4)}$$

POSSIBLE ROCs ...

- $|z| > .4$ $h[n]$ RIGHT-SIDED (POTENTIALLY CAUSAL) STABLE
- $|z| < .2$ $h[n]$ LEFT-SIDED, ~~NOT STABLE~~ NOT CAUSAL
- $.2 < |z| < .4$ $h[n]$ BOTH-SIDED, NOT STABLE NOT CAUSAL

IZT ...

$$H(z) = \frac{1-.3z^{-1}}{(1-.2z^{-1})(1+.4z^{-1})}$$

$M = 1$
$N = 2$

$$H(z) = \frac{A_1}{1-.2z^{-1}} + \frac{A_2}{1+.4z^{-1}}$$

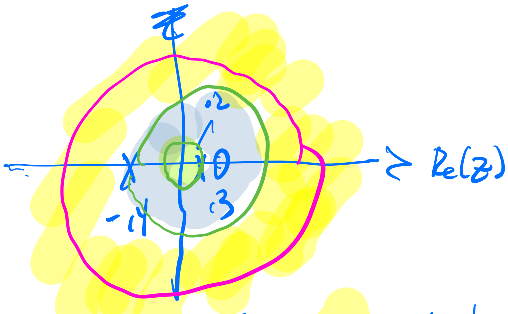
A_1, A_2 "RESIDUES"

$$A_i = H(z)(1-d_i z^{-1})\Big|_{z=d_i}$$

$$A_1 = \frac{(1-.3z^{-1})\cancel{(1-.2z^{-1})}}{\cancel{(1-.2z^{-1})}(1+.4z^{-1})}\Bigg|_{z=.2} = \frac{1-\frac{.3}{.2}}{1+\frac{.4}{.2}} = \frac{-1/2}{3} = -\frac{1}{6}$$

$$A_2 = \frac{(1-.3z^{-1})\cancel{(1+.4z^{-1})}}{(1-.2z^{-1})\cancel{(1+.4z^{-1})}}\Bigg|_{z=-.4} = \frac{1+\frac{.3}{.4}}{1+.2/.4} = \frac{7/4}{3/2} = \frac{7}{6}$$

$$H(z) = -\frac{1}{6}\frac{1}{(1-.2z^{-1})} + \frac{7}{6}\frac{1}{(1+.4z^{-1})}$$



$$\alpha^n u[n] \Longleftrightarrow \frac{1}{1-\alpha z^{-1}}, \quad |z|>|\alpha|$$

$$-\alpha^n u[-n-1] \Longleftrightarrow \frac{1}{1-\alpha z^{-1}}, \quad |z|<|\alpha|$$

ROC I $|z| > .4$ $h[n]$ RIGHT-SIDED (POTENTIALLY CAUSAL)
STABLE

ROC II $|z| < .2$ $h[n]$ LEFT-SIDED, NOT STABLE
NOT CAUSAL

ROC III $.2 < |z| < .4$ $h[n]$ BOTH-SIDED, NOT STABLE
NOT CAUSAL

ROC CASE I $h[n] = -\frac{1}{6}(.2)^n u[n] + \frac{7}{6}(-.4)^n u[n]$

ROC CASE II $h[n] = +\frac{1}{6}(.2)^n u[-n-1] - \frac{7}{6}(-.4)^n u[-n-1]$

ROC CASE III $h[n] = -\frac{1}{6}(.2)^n u[n] - \frac{7}{6}(-.4)^n u[-n-1]$

$$H(z) = \frac{1-.3z^{-1}}{(1-.2z^{-1})(1+.4z^{-1})} = \frac{1-.3z^{-1}}{1+.2z^{-1}-.08z^{-2}} = \frac{Y(z)}{X(z)}$$

$$Y(z)\cdot(1+.2z^{-1}-.08z^{-2}) = X(z)(1-.3z^{-1})$$

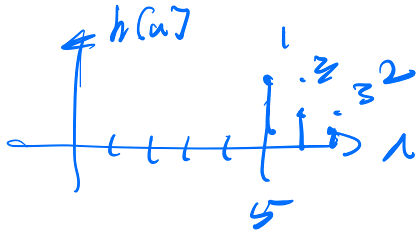
$$y[n]+.2y[n-1]-.08y[n-2] = x[n]-.3x[n-1]$$

$$y[n] = -.2y[n-1]+.08y[n-2]+x[n]-.3x[n-1]$$

CONSIDER

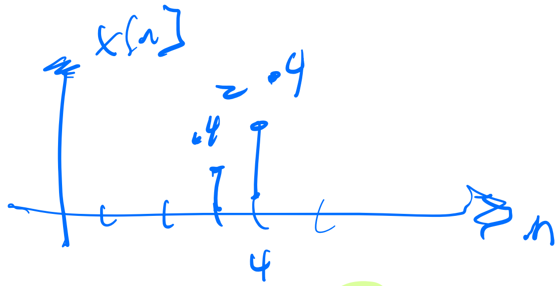
$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

$$h[n] = (.3)^{n-5}\, u[n-5]$$

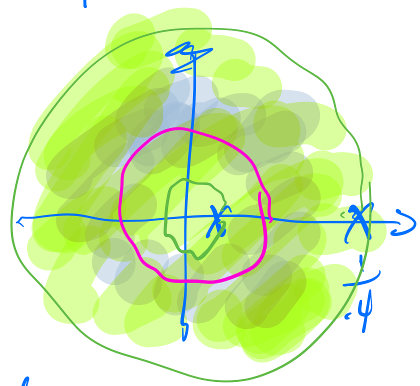


$$x[n] = (.4)^{-n+5}\, u[-n+4]$$

$$= \left(\frac{1}{.4}\right)^{n-5} u[-n+4]$$



$$H(z) = \frac{z^{-5}}{1-.3z^{-1}}, \quad |z| > .3$$



$$X(z) = \frac{z^{-5}}{1-\left(\frac{1}{.4}\right)z^{-1}}, \quad |z| < \frac{1}{.4}$$

$$Y(z) = z^{-10}\, \frac{1}{(1-.3z^{-1})}\, \frac{2}{\left(1-\left(\frac{1}{.4}\right)z^{-1}\right)} \qquad .3 < |z| < \frac{1}{.4}$$

$$y[n] = \left\{ A_1 (.3)^n u[n] + A_2 \left(\frac{1}{.4}\right)^n u[-n-1] \right\}$$

SHIFTED BY 10 SAMPLES TO THE RIGHT