1/26/24

RECITATION 1A

DTFTs

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

SYMMETRY PROPERTIES

$x[n] \Leftrightarrow X(e^{j\omega})$

$x[-n] \Leftrightarrow X(e^{-j\omega})$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[-n] e^{-j\omega n}$$

let $l = -n$
$n = -l$

$$= \sum_{l=-\infty}^{\infty} x[l] e^{-j\omega(-l)}$$

$$= \sum_{l=-\infty}^{\infty} x[l] e^{-j(-\omega)l}$$

$$= X(e^{-j\omega})$$

$x[n] = x[-n] \Leftrightarrow X(e^{j\omega}) = X(e^{-j\omega})$

EVEN $\Leftrightarrow$ EVEN

$x[n] = -x[-n] \Leftrightarrow X(e^{j\omega}) = -X(e^{-j\omega})$

ODD ODD

DTFT of $x^*[n]$

$$= \sum_{n=-\infty}^{\infty} x^*[n] e^{-j\omega n}$$

$$= \left(\sum_{n=-\infty}^{\infty} x[n] e^{+j\omega n}\right)^*$$

$$= \left(\sum_{n=-\infty}^{\infty} x[n] e^{-j(-\omega)n}\right)^*$$

$$= X^*(e^{-j\omega})$$

$x[n]$ REAL

$\Rightarrow x[n] = x^*[n] \Leftrightarrow X(e^{j\omega}) = X^*(e^{-j\omega})$

HERMITIAN SYMMETRY!

$$X(e^{j\omega}) = X_R(e^{j\omega}) + j\,X_I(e^{j\omega})$$

$$X_R(e^{j\omega}) = \mathrm{Re}\left[X(e^{j\omega})\right]$$
$$X_I(e^{j\omega}) = \mathrm{Im}\left[X(e^{j\omega})\right]$$

$x(n)$ REAL

$$X^*(e^{j\omega}) = X_R(e^{j\omega}) - j\,X_I(e^{j\omega})$$

$$X^*(e^{-j\omega}) = X_R(e^{-j\omega}) - j\,X_I(e^{-j\omega})$$

IF $x(n)$ REAL $\rightarrow$ $X_R(e^{j\omega}) = X_R(e^{-j\omega})$ $\quad X_R(e^{j\omega})$ EVEN

$X_I(e^{j\omega}) = -X_I(e^{-j\omega})$ $\quad X_I(e^{j\omega})$ ODD

| $x[n]$ | $\Longleftrightarrow$ $X(e^{j\omega})$ |
|---|---|
| EVEN | EVEN |
| ODD | ODD |
| REAL | $X(e^{j\omega}) = X^*(e^{-j\omega})$ $\quad X_R(e^{j\omega})$ EVEN, $X_I(e^{j\omega})$ ODD |
| REAL, EVEN | REAL, EVEN |
| REAL, ODD | IMAG, ODD |

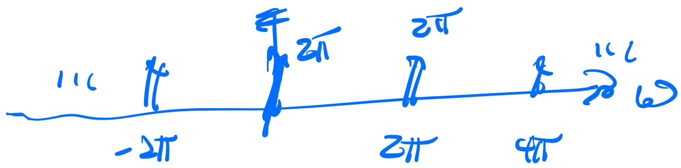
MORE EXAMPLES

DTFT of) $\cos(\omega_0 n)$

$$\sum_{r=-\infty}^{\infty} 2\pi\,\delta(\omega - 2\pi r)$$

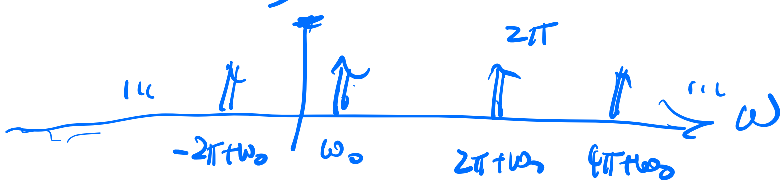
USING PROPERTIES ...

$1 \Longleftrightarrow$



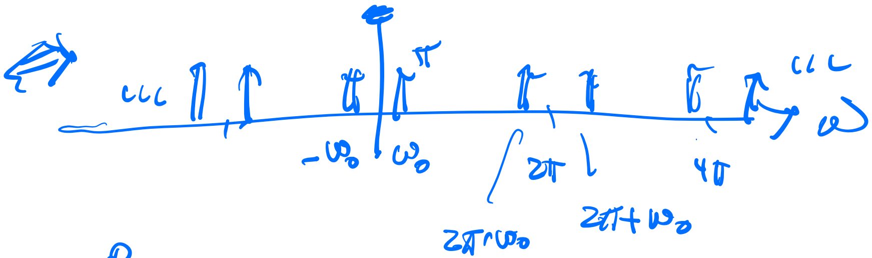

$$x(n)\cdot e^{j\omega_0 n} \Longleftrightarrow X(e^{j(\omega-\omega_0)})$$

$e^{j\omega_0 n} = 1\cdot e^{j\omega_0 n} \Longleftrightarrow$



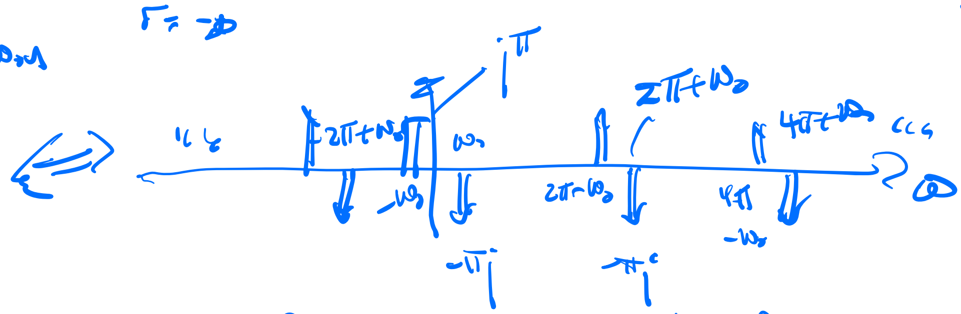

$$= 2\pi \sum_{r=-\infty}^{\infty} \delta(\omega - \omega_0 + 2\pi r)$$

$$\cos(\omega_0 n) = \frac{e^{j\omega_0 n} + e^{-j\omega_0 n}}{2} \Longleftrightarrow$$




$$= \pi \sum_{r=-\infty}^{\infty} \left( \delta(\omega - \omega_0 - 2\pi r) + \delta(\omega + \omega_0 - 2\pi r) \right)$$

$$\sin(\omega_0 n) = \frac{e^{j\omega_0 n} - e^{-j\omega_0 n}}{2j} \Longleftrightarrow$$




$$\frac{\pi}{j} = -\pi j$$

$$= j\pi \sum_{r=-\infty}^{\infty} -\delta(\omega - \omega_0 - 2\pi r) + \delta(\omega + \omega_0 - 2\pi r)$$

MORE PROPS

$$\boxed{\begin{aligned} &\star\quad x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega \\ &\star\quad X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} \end{aligned}}$$

PARSEVAL'S Th.

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2\, d\omega$$

ENERGY

ENERGY SPECTRAL DENSITY FUNCTION

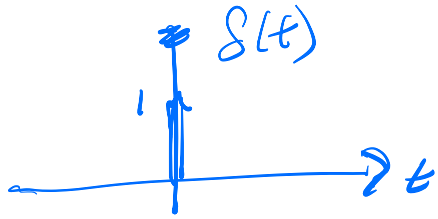
$$x[n]\Big|_{n=0} \Rightarrow x[0] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, \underbrace{\cancel{e^{j\omega 0}}}_{1}\, d\omega$$

$$x[0] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, d\omega$$

INITIAL VALUE THM

$$X(e^{j\omega})\Big|_{\omega=0} \qquad X(e^{j0}) = \sum_{n=-\infty}^{\infty} x[n]$$

# WORKING WITH DELTA FUNCTIONS

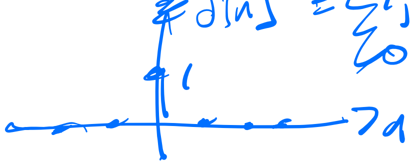



$\delta(t) = 0, \quad t \neq 0$

$\delta(0) = \infty$

$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$

$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$




$\underline{\text{EXPLICIT}}$

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$\delta(\cdot)$ $\underline{\text{IMPLICITLY}}$ DEFINED

$\phi(t)$ CONTINUOUS

$$\boxed{\int_{-\infty}^{\infty} \delta(t-a)\, \phi(t)\, dt = \phi(a)}$$

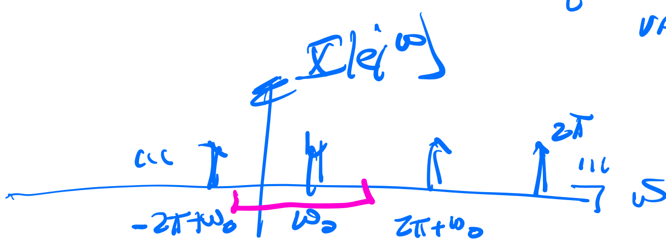
IMPLICIT DEF.

$\delta(t-a) = 0, \quad t \neq a$

$$\int_{-\infty}^{\infty} \delta(t)\, dt = \int_{-\infty}^{\infty} \delta(t) \cdot \underset{\phi(t)}{1}\, dt = 1$$

1. WHAT'S BEING INTEGRATED? $t$
2. WHAT VALUE of VAR INTEGRATED CAUSES VALUE of ( ) = 0 ? $(t=0)$
3. RESULT IS OTHER FUNCTION @ THAT VALUE of INTEGRATING VARIABLE

$e^{j\omega_0 n} \iff$




$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} 2\pi\, \delta(\omega - \omega_0)\, e^{j\omega n}\, d\omega$$

$$= e^{j\omega_0 n}$$

LINEARITY

SHIFT INVARIANCE

CAUSALITY

MEMORYLESSNESS

STABILITY

IF SYSTEM LSI

$\longrightarrow h[n] = 0, \; n < 0$

$\longrightarrow h[n] = 0, \; n < 0, \; n > 0$

$\longrightarrow \sum_{n=-\infty}^{\infty} |h[n]| < \infty$

EXAMPLES

1. $y[n] = x[n] \tan(\omega_0 n)$

$(a x_1[n] + b x_2[n]) \tan(\omega_0 n)$

LINEAR

NOT SI

CAUSAL

MEMORYLESS

NOT STABLE

2. $y[n] = \sum_{k=0}^{n} x[k]$

LINEAR

NOT SI

CAUSAL

NOT MEMORYLESS

NOT STABLE

$\sum_{k=0}^{n} x[k]$

SI $\sum_{k=n-n_0}^{n+n_0} x[k]$

PERIODICITY

$\cos\left(\frac{2\pi n}{N}\right), \; \forall n$

$N$ INTEGER $\rightarrow$ PERIODIC, PERIOD $N$

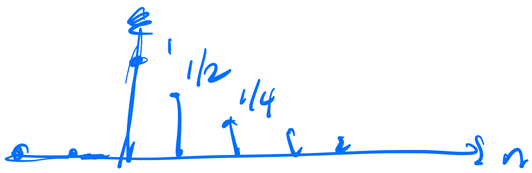
$N$ RATIONAL FRACTION $\rightarrow$ PERIODIC, PERIOD $\neq N$

$N$ IRRATIONAL $\rightarrow$ NOT PERIODIC

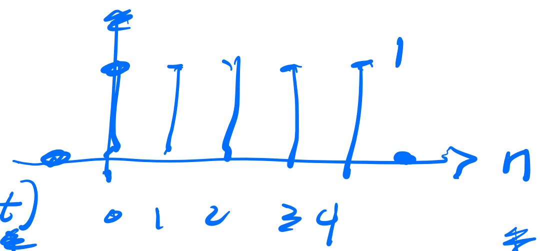
NOT $\Sigma$ $\cos(\omega_0 n) u[n]$

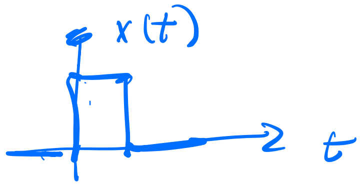
# CONVOLUTION EXAMPLES

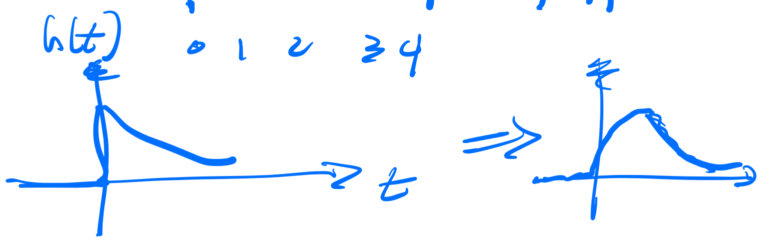
$x[n] = \left(\frac{1}{2}\right)^n u[n]$



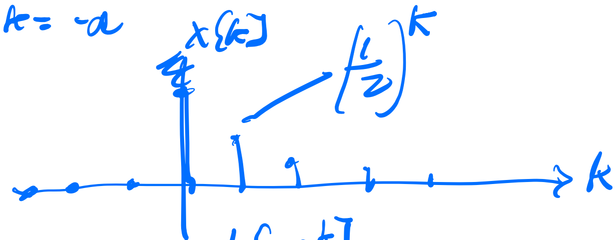

$h[n] = u[n] - u[n-5]$

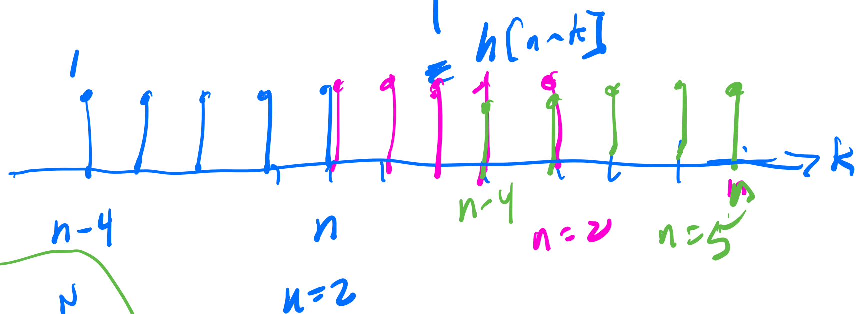



$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$




$$\boxed{\sum_{n=0}^{N-1} \alpha^n = \frac{1-\alpha^N}{1-\alpha}}$$

FOR $n<0$, $y[n]=0$

FOR $0 \le n \le 4$

$$y[n] = \sum_{k=0}^{n} \left(\frac{1}{2}\right)^k = \frac{1-\left(\frac{1}{2}\right)^{n+1}}{1-\frac{1}{2}}$$

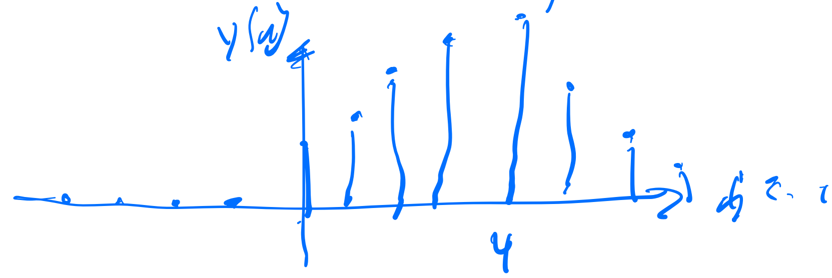
$$= 2 - \left(\frac{1}{2}\right)^n$$

FOR $n \ge 4$ $\quad y[n] = \sum_{k=n-4}^{n} \left(\frac{1}{2}\right)^k = \sum_{\ell=0}^{4} \left(\frac{1}{2}\right)^{\ell+n-4}$

Let $\ell = k-(n-4) = k-n+4$

$k = \ell + n - 4$

$$y[n] = \left(\frac{1}{2}\right)^{n-4} \sum_{\ell=0}^{4} \left(\frac{1}{2}\right)^{\ell} = \left(\frac{1}{2}\right)^{n-4} \left(\frac{1-\left(\frac{1}{2}\right)^5}{1-\frac{1}{2}}\right) = \left(\frac{1}{2}\right)^{n-4} \left(2 - \left(\frac{1}{2}\right)^4\right)$$




Ex. $\cos(\omega_0 n) \cdot u[n]$

1. $\cos(\omega_0 n) = \dfrac{e^{j\omega_0 n} + e^{-j\omega_0 n}}{2}$

2. BALANCING EXPS $\qquad \dfrac{1 - e^{-j\omega N}}{1 - e^{-j\omega}}$

$$= \frac{e^{-j\frac{\omega N}{2}}\left(e^{j\frac{\omega N}{2}} - e^{-j\frac{\omega N}{2}}\right)}{e^{-j\frac{\omega}{2}}\left(e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}\right)}$$

3. COMBINING TERMS