AN INTRODUCTION TO 2-DIMENSIONAL DSP

Richard M. Stern

Department of Electrical and Computer Engineering,
School of Computer Science,
and Biomedical Engineering Program
Carnegie Mellon University
Pittsburgh, Pennsylvania 15213

Telephone: (412) 268-2535
FAX: (412) 268-3890
INTERNET: `rms@cs.cmu.edu`

INTRODUCTION

Background: Many types of analyses make use of 2-dimensional images ....

- Photographs
- Satellite images
- X-rays and other medical images

Many concepts from 1-D DSP are directly extensible to two dimensions, but some are not

Goals of this lecture:

- To summarize basic 2-D relationships
- To identify which concepts do or do not extend to 2-D
- To briefly discuss 2-D filter design approaches

For further reading:

- *Image Processing* by Jae Lim (and many other texts)

SOME EXAMPLES OF 2-D SIGNALS

The unit sample function:

$$\delta[n_1, n_2] = 1, n_1 = n_2 = 0; \;\; 0, \text{otherwise}$$

The unit step function:

$$u[n_1, n_2] = 1, n_1 \geq 0, n_2 \geq 0$$

The exponential function:

$$x[n_1, n_2] = a^{n_1} b^{n_2}$$

Cosine function:

$$x[n_1, n_2] = \cos(\omega_1 n_1 + \phi_1)\cos(\omega_2 n_2 + \phi_2)$$

Note: A sequence is **separable** if

$$x[n_1, n_2] = x[n_1]x[n_2]$$

2-D LSI SYSTEMS

$$x[n_1,n_2] \longrightarrow \boxed{T} \longrightarrow y[n_1,n_2]$$

A system is **linear** if

$$ax[n_1, n_2] + bx[n_1, n_2] \Rightarrow ay[n_1, n_2] + by[n_1, n_2]$$

A system is **shift invariant** if for all k,l,

$$x[n_1 - k, n_2 - l] \Rightarrow y[n_1 - k, n_2 - l]$$

If a 2-D system is LSI, then

$$\delta[n_1, n_2] \Rightarrow h[n_1, n_2]$$

THE CONVOLUTION SUM

As in 1-D, we can represent an input as 2-D shifted and scaled delta functions producing ....

1-D convolution:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

2-D convolution:

$$y[n_1, n_2] = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} x[k_1, k_2]h[n_1-k_1, n_2-k_2]$$

Note that if both x and h are separable, then

$$y[n_1, n_2] = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} x[k_1]x[k_2]h[n_1-k_1]h[n_2-k_2]$$

or ....

$$y[n_1, n_2] = \sum_{k_1=-\infty}^{\infty} x[k_1]h[n_1-k_1] \sum_{k_2=-\infty}^{\infty} x[k_2]h[n_2-k_2]$$

In other words, if x and h are separable, the 2-D convolution degenerates into the product of two 1-D convolutions.

SOME SYSTEM PROPERTIES

A system is **causal** if $h[n_1, n_2] = h[n_1, n_2]u[n_1, n_2]$

(although this is not a big deal in 2-D)

A system is **stable** if $\sum_{n_1 = -\infty}^{\infty} \sum_{n_2 = -\infty}^{\infty} |h[n_1, n_2]| < \infty$

Difference equations for causal 1-D systems:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{r=0}^{M} b_r x[n-r]$$

Difference equations for causal 2-D systems:

$$\sum_{k_1=0}^{N_1} \sum_{k_2=0}^{N_2} a_{k_1, k_2} y[n_1 - k_1, n_2 - k_2] =$$

$$= \sum_{r_1=0}^{M_1} \sum_{r_2=0}^{M_2} b_{r_1, r_2} x[n_1 - r_1, n_2 - r_2]$$

THE 2-D DTFT

In 1-D we have

$$e^{j\omega n} \longrightarrow \boxed{\;T\;} \longrightarrow H(e^{j\omega n})e^{j\omega n}$$

In 2-D we have

$$e^{j\omega 2n1}e^{j\omega 2n2} \longrightarrow \boxed{\;T\;} \longrightarrow H(e^{j\omega 1n1}, e^{j\omega 2n2})e^{j\omega 1n1}e^{j\omega 2n2}$$

From the convolution sum definition we can obtain

$$H\left(e^{j\omega_1}, e^{j\omega_2}\right) = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} h[n_1, n_2] e^{-j\omega_1 n_1} e^{-j\omega_2 n_2}$$

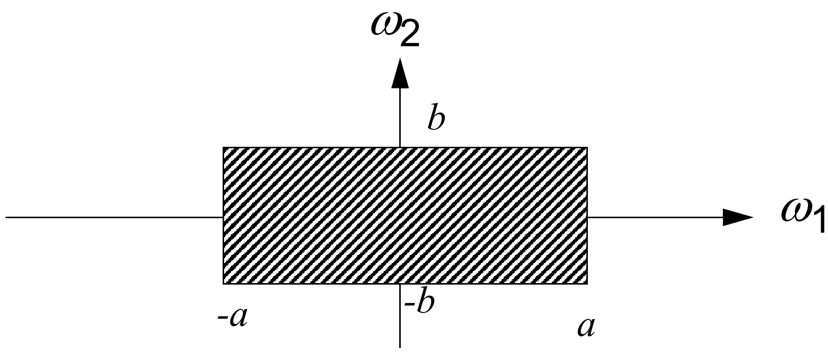
and

$$h[n_1, n_2] = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H\left(e^{j\omega_1}, e^{j\omega_2}\right) e^{j\omega_1 n_1} e^{j\omega_2 n_2} d\omega_1 d\omega_2$$

Comments:

- $H\left(e^{j\omega_1}, e^{j\omega_2}\right)$ is periodic in ω_1 and ω_2
- If $h[n_1, n_2]$ is separable, $H\left(e^{j\omega_1}, e^{j\omega_2}\right)$ is as well, and computing the 2-D DTFT becomes just a matter of computng the product of two 1-D DTFTs
- Convolution in time $\Leftrightarrow$ multiplication in frequency

SOME EXAMPLES OF FREQUENCY RESPONSE

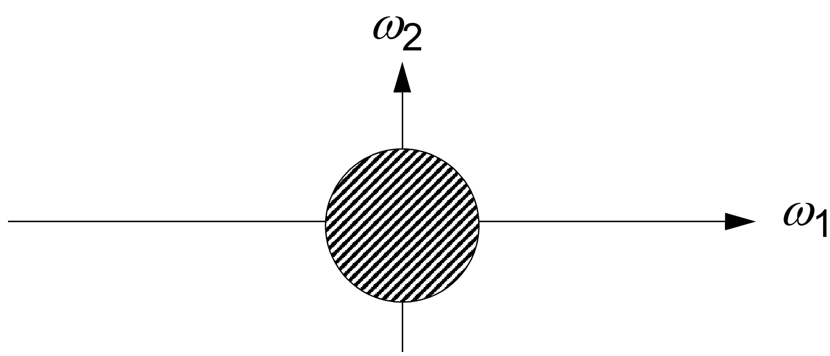


$$H\left(e^{j\omega_1}, e^{j\omega_2}\right) = 1, |\omega_1| \le a, |\omega_2| \le b$$

This DTFT is **separable**, and

$$h[n_1, n_2] = \frac{\sin(an_1)}{\pi n_1}\frac{\sin(bn_2)}{\pi n_2}$$

A SECOND FREQUENCY RESPONSE



$$H\left(e^{j\omega_1}, e^{j\omega_2}\right) = 1,\ \omega_1^2 + \omega_2^2 \le R^2; 0 \text{ otherwise}$$

This DTFT is **not separable!**

In fact, it can be shown that

$$h[n_1, n_2] = \frac{\omega_c}{2\pi\sqrt{n_1^2 + n_2^2}} J_1\left(\omega_c \sqrt{n_1^2 + n_2^2}\right)$$

Note: Even though this function is not separable, it is **rotation-invariant** in both time and frequency.

2-D Z-TRANSFORMS

In a similar fashion to the 1-D case, we build up Z-transforms by modeling time functions as linear combinations of the function

$$z_1^{n_1} z_2^{n_2} = \left(r_1 e^{j\omega_1}\right)^{n_1} \left(r_2 e^{j\omega_2}\right)^{n_2}$$

In particular,

$$H(z_1, z_2) = \sum_{n_1 = \infty}^{\infty} \sum_{n_2 = -\infty}^{\infty} h[n_1, n_2] z_1^{-n_1} z_2^{-n_2}$$

and

$$h[n_1, n_2] = \frac{1}{(2\pi j)^2} \int_{C_1} \int_{C_2} H(z_1, z_2) z_1^{n_1 - 1} z_2^{n_2 - 1} dz_1 dz_2$$

Comments: No poles and zeros (!), so

- No easy tests for stability
- No parallel or cascade implementations
- No Parks-McClellan algorithm
- etc etc

THE 2-D DFT

The 2-D DFT is derived in a fashion similar to how it had been in 1-D ... the definitions:

$$H[k_1, k_2] = \sum_{n_1 = 0}^{N_1 - 1} \sum_{n_2 = 0}^{N_2 - 1} h[n_1, n_2] W_{N_1}^{k_1 n_1} W_{N_2}^{k_2 n_2}$$

and

$$h[n_1, n_2] = \frac{1}{N_1 N_2} \sum_{k_1 = 0}^{N_1 - 1} \sum_{k_2 = 0}^{N_2 - 1} H[k_1, k_2] W_{N_1}^{-k_1 n_1} W_{N_2}^{-k_2 n_2}$$

Comments:

- Multiplying DFT coefficients in frequency corresponds to a 2-D circular (“toroidal”) convolution
- Overlap-add, overlap-save algorithms still valid

COMPUTING THE 2-D DFT

Recall that the 2-D DFT is ...

$$H[k_1, k_2] = \sum_{n_1 = 0}^{N_1 - 1} \sum_{n_2 = 0}^{N_2 - 1} h[n_1, n_2] W_{N_1}^{k_1 n_1} W_{N_2}^{k_2 n_2}$$

We can rewrite $H[k_1, k_2]$ as

$$H[k_1, k_2] = \sum_{n_1 = 0}^{N_1 - 1} W_{N_1}^{k_1 n_1} \sum_{n_2 = 0}^{N_2 - 1} h[n_1, n_2] W_{N_2}^{k_2 n_2}$$

Let

$$\sum_{n_2 = 0}^{N_2 - 1} h[n_1, n_2] W_{N_2}^{k_2 n_2} \equiv g[n_1, k_2]$$

then, $H[k_1, k_2] = \sum_{n_1 = 0}^{N_1 - 1} g[n_1, k_2] W_{N_1}^{k_1 n_1}$

So .... to compute the 2-D DFT,

- Take 1-D DFT of each column
- Take the row-wise DFTs of the resulting coefficients

SOME SUMMARY OBSERVATIONS ABOUT 2-D DSP ....

A few things that are obvious extensions of 1-D DSP:

- Linearity and shift invariance
- Convolution sum
- 2-D DTFTs
- 2-D DFTs

Some things that are different:

- 2-D Z-transforms (no poles or zeros!)

Some other things to keep in mind:

- Tradeoff between separability and rotation invariance
- Physical significance of 2-D complex exponentials
- Efficiencies provided by separability
- Efficient computation of 2-D DFT

Next topic of discussion:

- 2-D discrete-time filter design