5.6.1 Frequency-Sampling Design

In Chapter 3 we showed that a finite-duration sequence can be represented by its discrete Fourier transform. Thus an FIR filter has a representation in terms of the "frequency samples"

$$\tilde{H}(k) = H(z)\big|_{z=e^{j(2\pi/N)k}} = \sum_{n=0}^{N-1} h(n)e^{-j(2\pi/N)kn}, \qquad k = 0, 1, \ldots, N-1$$

As we showed in Chapter 3, $H(z)$ can be represented in terms of the samples $\tilde{H}(k)$ by the expression

$$H(z) = \frac{1 - z^{-N}}{N} \sum_{k=0}^{N-1} \frac{\tilde{H}(k)}{1 - e^{j(2\pi/N)k}z^{-1}} \tag{5.56}$$

As was shown in Chapter 4, Eq. (5.56) serves as the basis of the frequency-sampling realization of an FIR filter. If we let $z = e^{j\omega}$, then the frequency response has the representation

$$\begin{aligned} H(e^{j\omega}) &= \frac{1 - e^{-j\omega N}}{N} \sum_{k=0}^{N-1} \frac{\tilde{H}(k)}{1 - e^{j(2\pi/N)k}e^{-j\omega}} \\ &= \frac{e^{-j\omega((N-1)/2)}}{N} \sum_{k=0}^{N-1} \tilde{H}(k)e^{j\pi k(1-1/N)} \frac{\sin[N(\omega - (2\pi/N)k)/2]}{\sin[(\omega - (2\pi/N)k)/2]} \end{aligned} \tag{5.57}$$

Equation (5.57) suggests a simple but rather naive approach to filter design, i.e., to specify the filter in terms of samples of one period of the desired frequency response

$$\tilde{H}(k) = H_d(e^{j(2\pi/N)k}), \qquad k = 0, 1, \ldots, N-1$$

relying on the interpolation indicated in Eq. (5.57) to "fill in the gaps" in the frequency response. As an illustration of this approach, consider the approximation of an ideal lowpass filter with cutoff frequency $\omega_c = \pi/2$. Figure 5.39(a) shows the desired frequency response $H_d(e^{j\omega})$ and the samples $\tilde{H}(k)$ for $N = 33$. As can be seen, the magnitude of the frequency response is specified at multiples of $2\pi/33$ radians, with the cutoff frequency $\omega_c = \pi/2$ being between $\omega = 16\pi/33$ and $18\pi/33$. The phase is taken to be linear

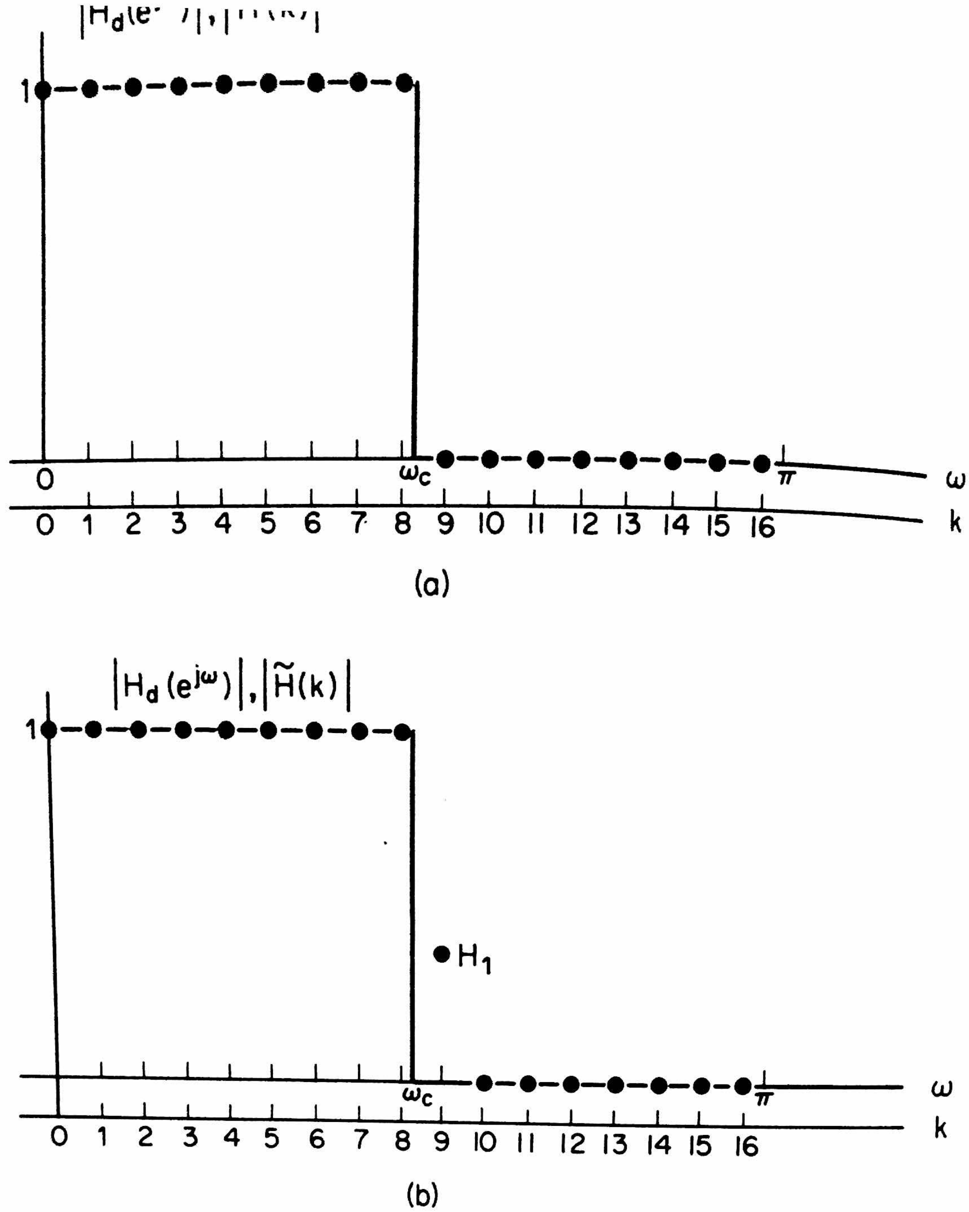

Fig. 5.39 Fixed samples of ideal lowpass filter frequency response: (a) no transition sample; (b) one transition sample H_1.

with delay equal to $(N-1)/2$ samples. The impulse response can of course be obtained using the inverse discrete Fourier transform, as in

$$h(n) = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{H}(k)e^{j(2\pi/N)kn} \qquad n = 0, 1, \ldots, N-1 \qquad (5.58)$$

$$= 0 \qquad \text{otherwise}$$

If we evaluate the frequency response corresponding to such a filter, we obtain the rather disappointing curve shown in Fig. 5.40(a). This figure shows $20 \log_{10} |H(e^{j\omega})|$, with the fixed sample points being indicated by the heavy dots in the passband and the points indicating infinite attenuation at the zero samples in the stopband. We note that there is a smooth transition between $16\pi/33$ and $18\pi/33$; however, the minimum stopband attenuation is somewhat less than 20 dB. This filter would be unsatisfactory for most purposes. As we have repeatedly seen, one way to improve the stopband attenuation is to widen the transition band. This can be easily done in this

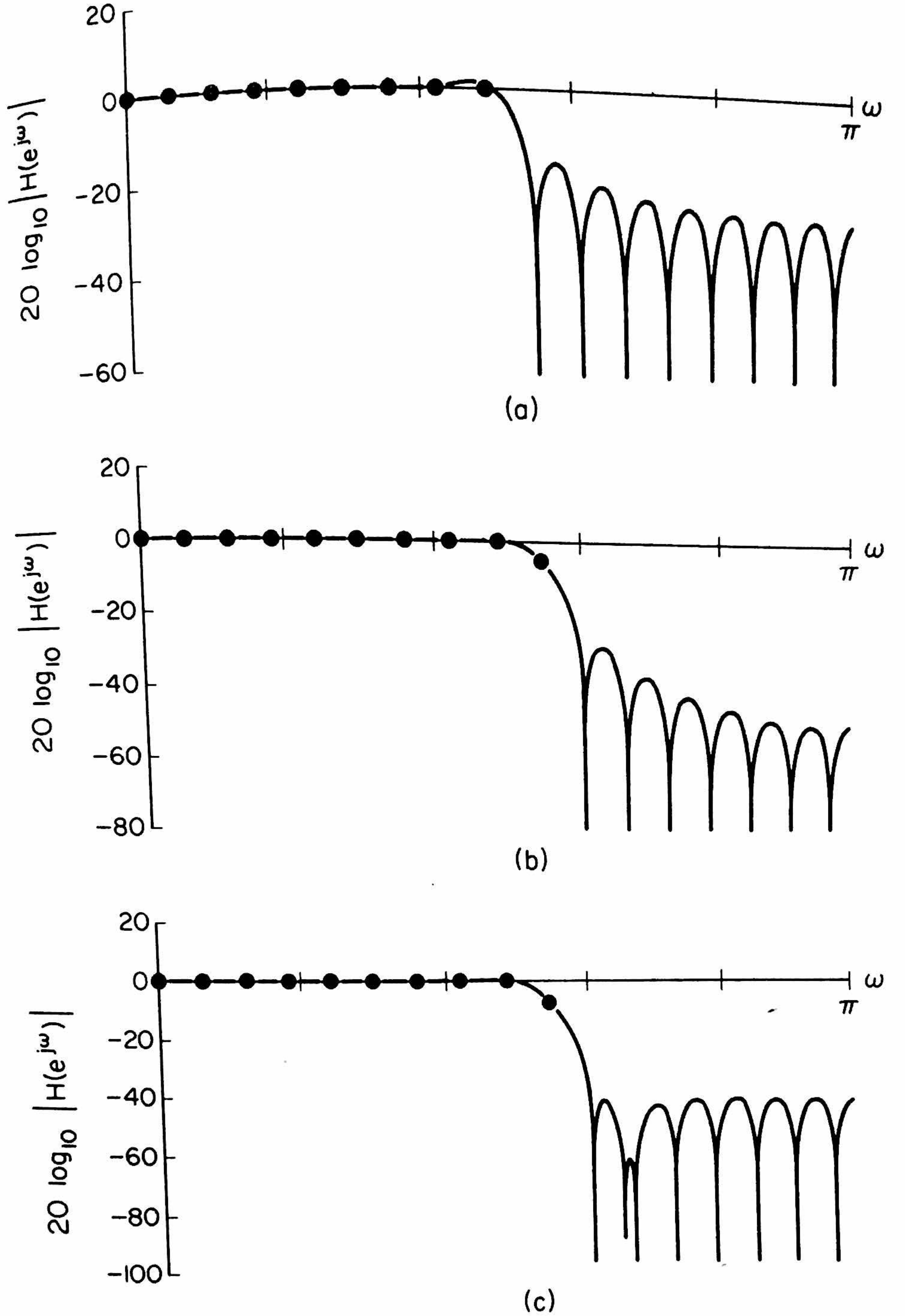

Fig. 5.40 Effect of a single transition sample: (a) $H_1 = 0$ (no transition sample); (b) $H_1 = 0.5$; (c) $H_1 = 0.3904$.

case by allowing a sample at the boundary between passband and stopband to take on a value different from either 1 or 0, as depicted in Fig. 5.39(b). Figure 5.40(b) shows the frequency response for $H_1 = 0.5$. Note that the transition band is now about twice as wide, but the minimum stopband attenuation is considerably greater.

It can be seen from Eq. (5.57) that $H(e^{j\omega})$ is a linear function of the parameters $\tilde{H}(k)$. Thus linear optimization techniques can be used to vary these parameters so as to give the best approximation to the desired filter. This approach, first proposed by Gold and Jordan [23], and developed by Rabiner, et al. [24], has been used to design a variety of filters. For example,

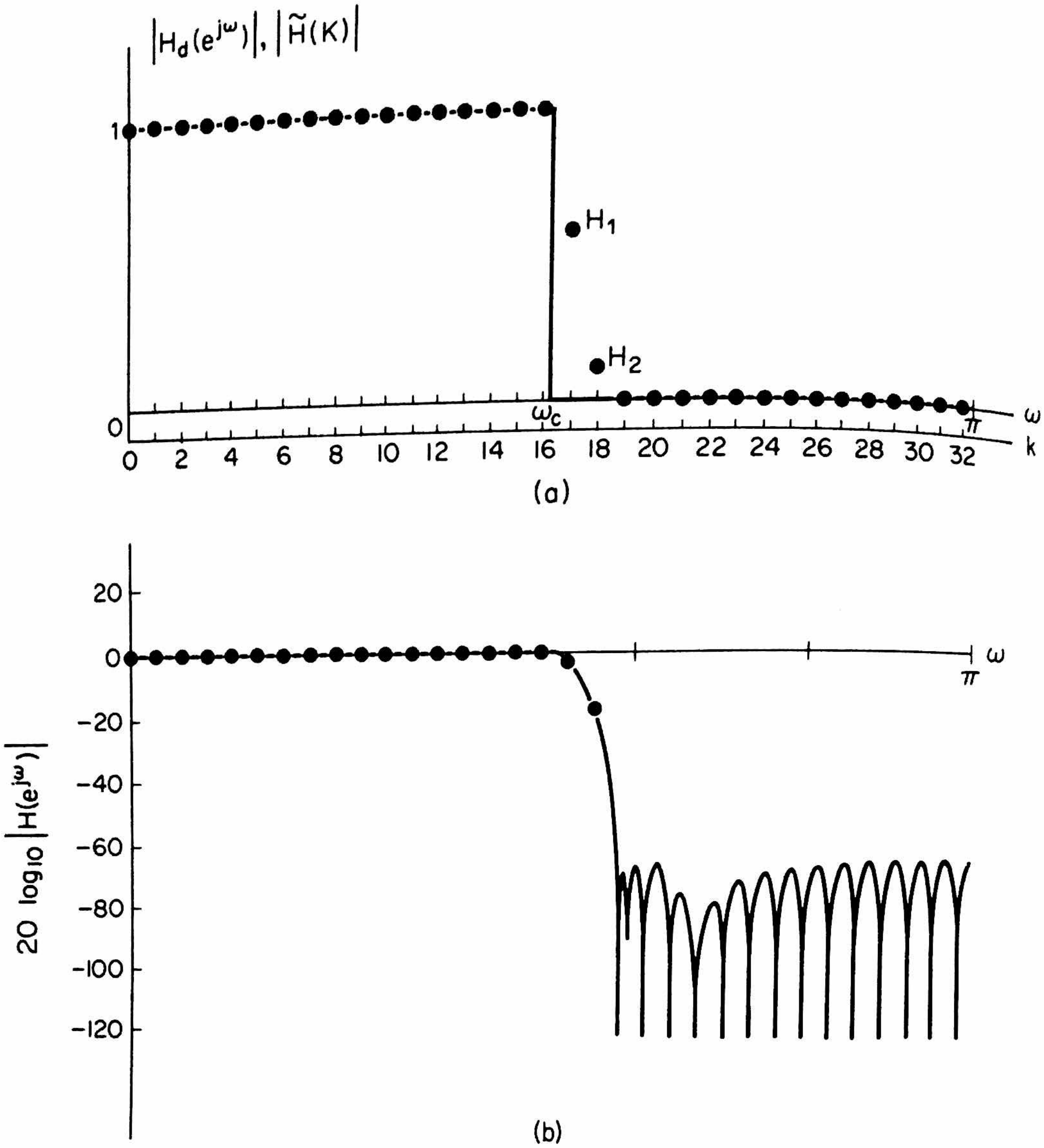

Fig. 5.41 Frequency sampling design using two transition samples: (a) desired frequency response with fixed samples and two transition samples; (b) resulting optimum frequency response for two transition samples.

in the case that we are discussing, a simple gradient search technique can be used to choose the value of H_1 such that the maximum error in either the passband or stopband is minimized. Figure 5.40(c) shows the response for $H_1 = 0.3904$, the value that minimizes the error (maximizes the attenuation) in the stopband

$$\frac{20\pi}{33} \leq |\omega| \leq \pi$$

Thus, it is clear that the stopband attenuation is significantly improved. If further improvement is required, we can broaden the transition region further by allowing a second† sample to differ from 1 or 0. If N is held fixed, this

† Rabiner et al. [24] give results for lowpass filters with up to four variable transition samples.

results in a transition region twice as wide. However, greater attenuation can be achieved. Of course, if we double N, the transition width remains the same, while allowing two transition samples to vary. Figure 5.41(a)shows such a set of samples for the example that we have been discussing for $N = 65$.† Figure 5.41(b) shows $20 \log_{10} |H(e^{j\omega})|$ for $N = 65$ and

$$H_1 = \tilde{H}(17) = H(e^{j(34\pi/65)}) = 0.5886$$

$$H_2 = \tilde{H}(18) = H(e^{j(36\pi/65)}) = 0.1065$$

These are very close to the optimum transition samples that minimize the maximum absolute error (maximize the attenuation) in the stopband. As can be seen, by comparing Fig. 5.41(b) to Fig. 5.40(c), by using two transition samples and increasing N by approximately a factor of 2 (from 33 to 65), the stopband attenuation is increased by about 24 dB, with a transition band that is somewhat narrower ($6\pi/65$ versus $8\pi/66$) than for one transition sample when $N = 33$.

Frequency sampling designs are particularly attractive for narrow-band frequency selective filters where only a few of the samples of the frequency response are nonzero [25,26]. In such cases a frequency sampling realization as discussed in Chapter 4 may be considerably more efficient than either direct convolution or convolution using the DFT. In general, even if more than a few samples are nonzero, the frequency-sampling design method yields excellent results. However, it is clear from the example of lowpass filter design that the method lacks flexibility in specifying the passband and stopband cutoff frequencies since the placement of ones and zeros and transition samples is constrained to integer multiples of $2\pi/N$. By making N large enough, samples can be obtained arbitrarily close to any given frequency; however, this is an inefficient approach. For this reason, particularly if the filter is not to be realized using the frequency sampling structure, other algorithmic design techniques have been developed with more attractive features for general frequency selective filter design.