IIR Filter Design Examples

(OSYP 7.2.1)

Review:

Impulse Invariance

$$h[n] = T h_c(nT)$$

$$H(s) = \frac{1}{s+a} \Rightarrow H(z) = \frac{T}{1 - e^{-aT} z^{-1}}$$

$$H(s) = \sum_{k=1}^{N} \frac{A_k}{s - s_k} \Rightarrow H(z) = \sum_{k=1}^{N} \frac{A_k T z}{z - e^{-s_k T}}$$

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} H\left(j\left(\Omega - k\frac{2\pi}{T}\right)\right)$$

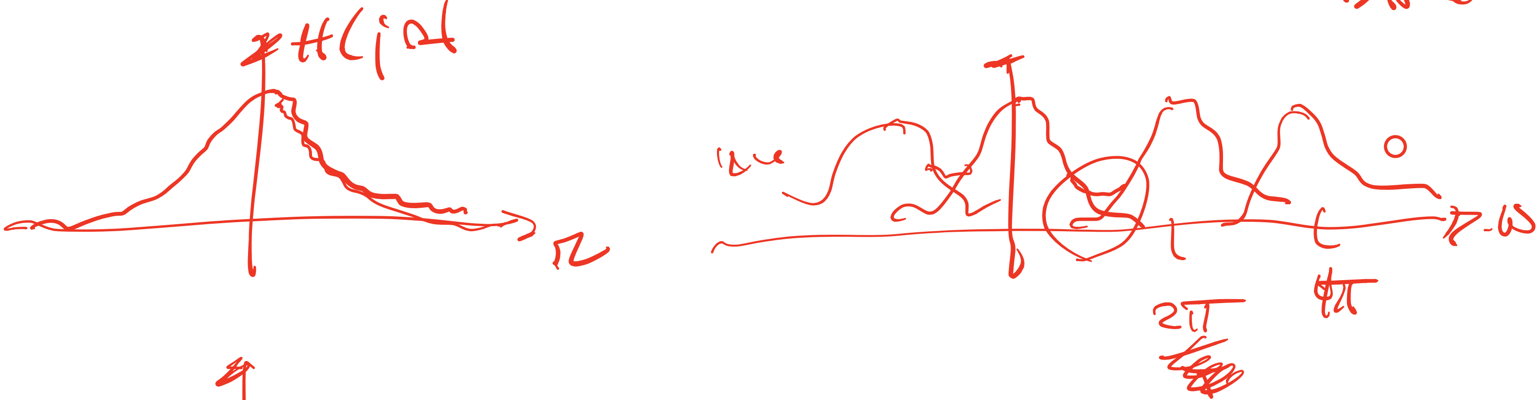
Bilinear Transform

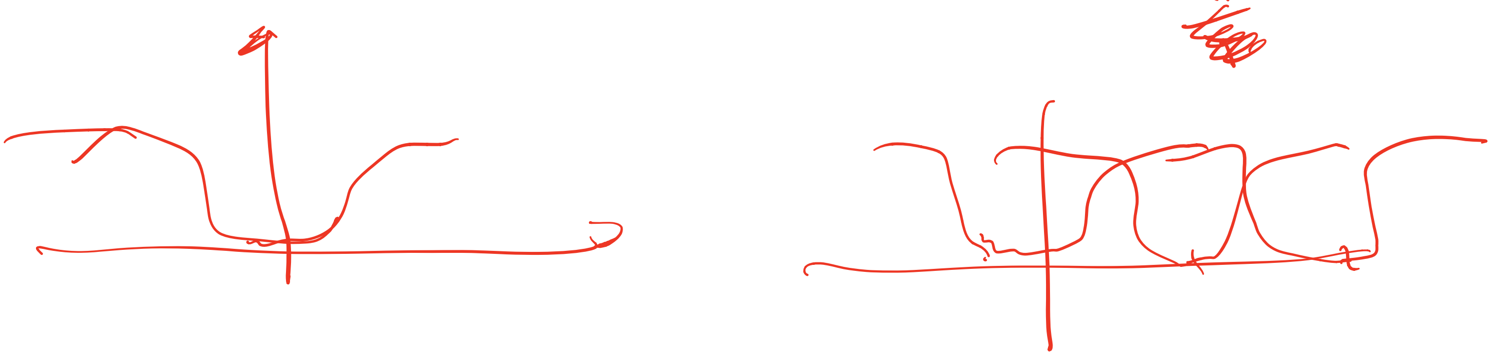
$$s = \frac{2}{T} \frac{1 - z^{-1}}{1 + z^{-1}}$$

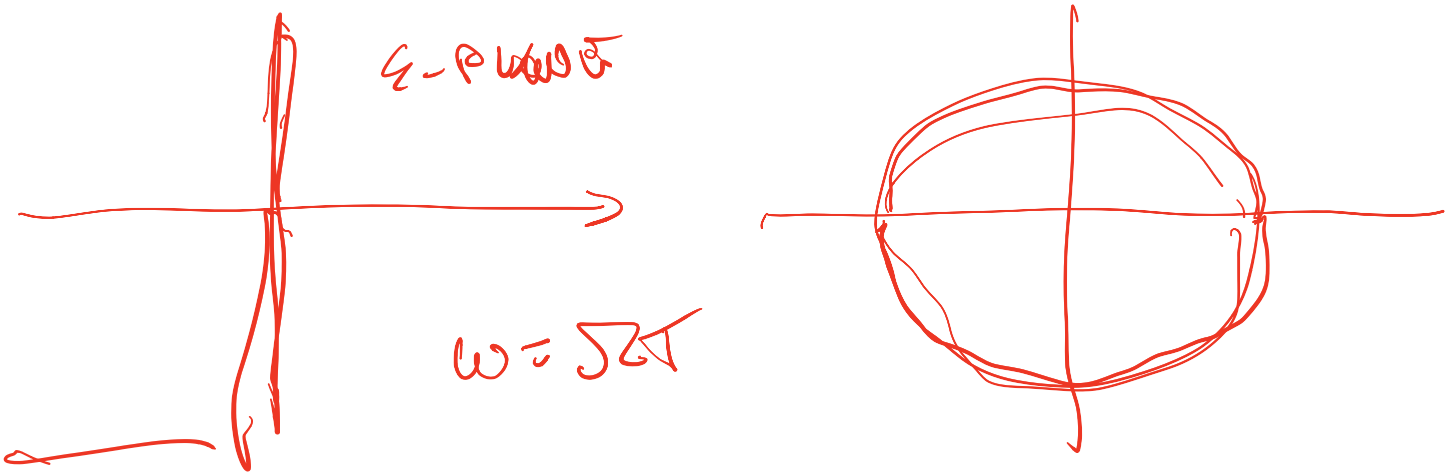
$$H(e^{j\omega}) = H(j\Omega)\Big|_{\Omega = \frac{2}{T}\tan\left(\frac{\omega}{2}\right)}$$

OR $\omega = 2\tan^{-1}\left(\frac{\Omega T}{2}\right)$

ALIASING WITH IMPULSE INVARIANCE







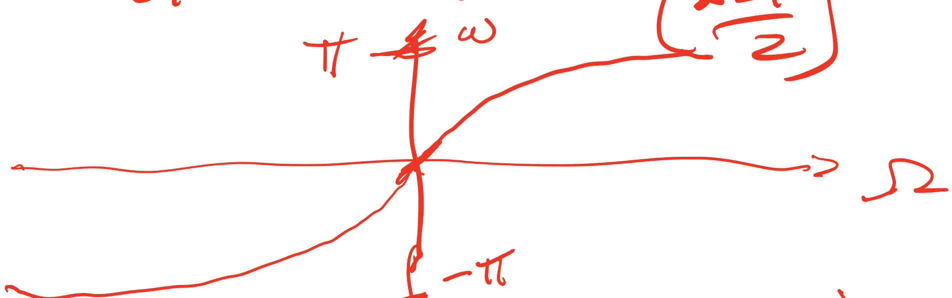
$\omega = \Omega T$

BILINEAR TRANSFORM

$$\to s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$$

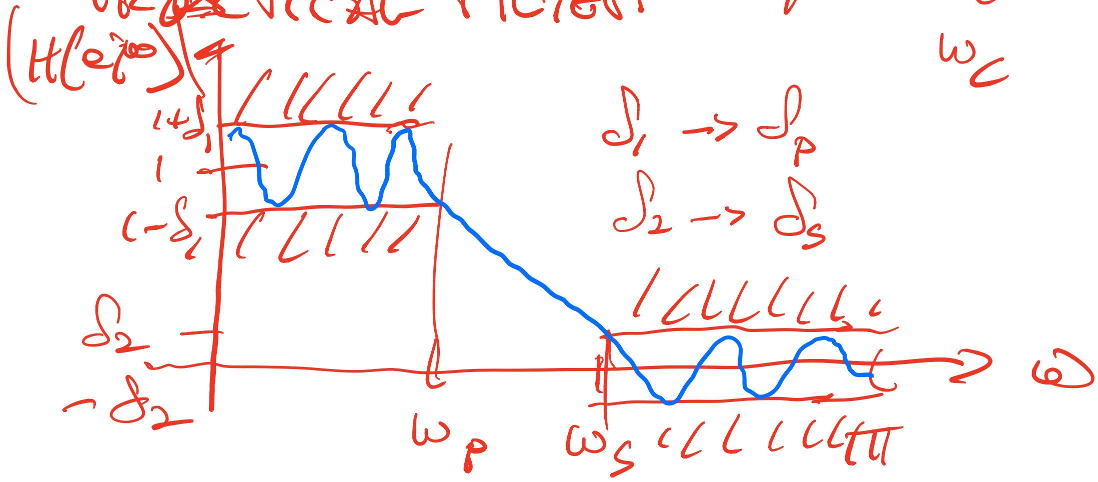
$$H(e^{j\omega}) = H(j\Omega)\Big|_{\Omega = \frac{2}{T}\tan\left(\frac{\omega}{2}\right)}$$

OR $\omega = 2\tan^{-1}\left(\frac{\Omega T}{2}\right)$



IDEAL LPF →

PRACTICAL FILTER

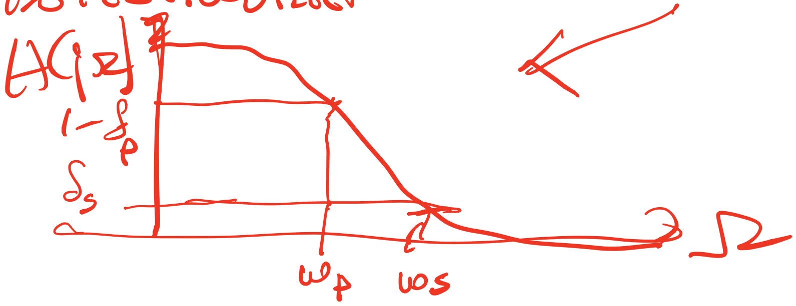


$\delta_1 \to \delta_p$

$\delta_2 \to \delta_s$

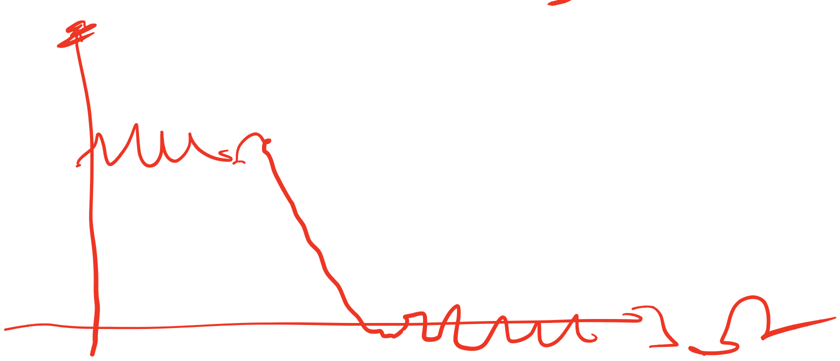
PRACTICAL IIR CT FILTERS

BUTTERWORTH



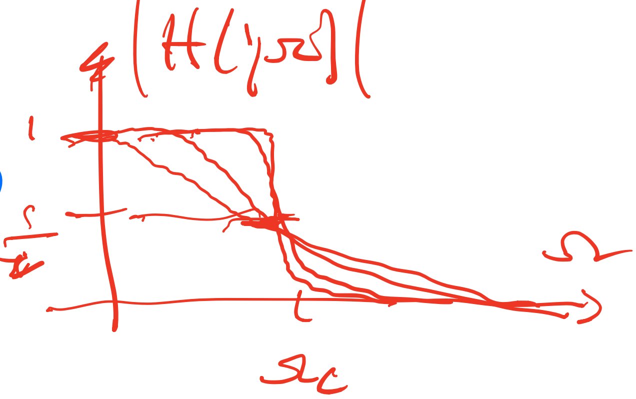
CHEBYSHEV FILTERS

ELLIPTICAL (CAUER)

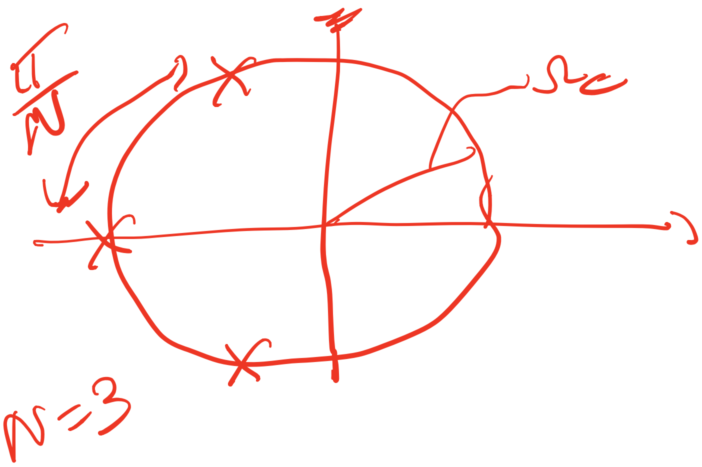


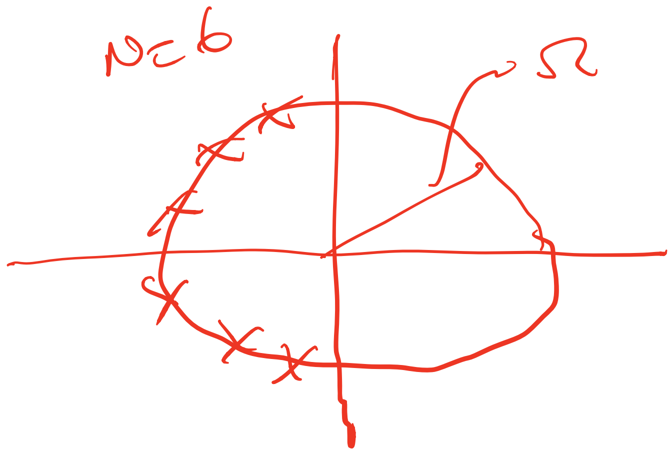
Butterworth Filter

$$|H(j\Omega)|^2 = \frac{1}{1+\left(\frac{j\Omega}{j\Omega_c}\right)^{2N}}$$



Pole locations





Specs of DT LPF

$$\omega_p = 0.15\pi \qquad \Delta\omega = .2\pi$$
$$\omega_s = .35\pi$$

Passband Ripple $\quad -3\text{dB} \le |H(e^{j\omega})| \le 0\text{dB}, \quad 0 \le |\omega| \le \omega_p$

Stopband Ripple $\quad -20\text{dB} \ge |H(e^{j\omega})|, \quad \omega_s \le |\omega| \le \pi$

$$x\,\text{dB} = 20\log_{10}(x)$$

$$\frac{x\,\text{dB}}{20} = \log_{10}(x)$$

$$10^{x\text{dB}/20} = x$$

$$1-\delta_p = 10^{-3/20} = .7079$$

$$\delta_s = 10^{-20/20} = 0.1$$

DEFINE $k_1 \triangleq \frac{1}{(1-\delta_p)^2} - 1 = .9953$

$$k_2 = \frac{1}{\delta_s^2} - 1 = 99$$

$$|H(j\Omega)|^2 = \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}}$$

MEETING SPEC @ PASSBAND $\Rightarrow \frac{1}{1+\left(\frac{\Omega_p}{\Omega_c}\right)^{2N}} = (1-\delta_p)^2$

$$1+\left(\frac{\Omega_p}{\Omega_c}\right)^{2N} = \frac{1}{(1+\delta_p)^2}$$

$$\boxed{\left(\frac{\Omega_p}{\Omega_c}\right)^{2N} = \frac{1}{(1+\delta_p)^2} - 1 = k_1}$$

MEETING SPECS @ STOPBAND

$$\frac{1}{1+\left(\frac{\Omega_s}{\Omega_c}\right)^{2N}} = \delta_s^2 \Rightarrow \boxed{\left(\frac{\Omega_s}{\Omega_c}\right)^{2N} = \frac{1}{\delta_s^2} - 1 = k_2}$$

$$\left(\frac{\Omega_p}{\Omega_s}\right)^{2N} = \frac{\left(\frac{\Omega_p}{\Omega_c}\right)^{2N}}{\left(\frac{\Omega_s}{\Omega_c}\right)^{2N}} = \frac{k_1}{k_2} \; ; \quad 2N \log\left(\frac{\Omega_p}{\Omega_s}\right) = \log\left(\frac{k_1}{k_2}\right)$$

$$N = \frac{\log\left(\frac{k_1}{k_2}\right)}{2\log\left(\frac{\Omega_p}{\Omega_s}\right)}$$

$$N = \frac{\log\left(k_2/k_1\right)}{2\log\left(\Omega_s/\Omega_p\right)}$$

$$\frac{\log(a)}{\log(b)} = \frac{\log\left(\frac{1}{a}\right)}{\log\left(\frac{1}{b}\right)} = \frac{-\log(a)}{-\log(b)}$$

$\Rightarrow N = \underline{\underline{2.7144}} \rightarrow 3$ POLES

IMPULSE INVARIANCE

$$\omega = \Omega T \; ; \; \Omega = \frac{\omega}{T}$$

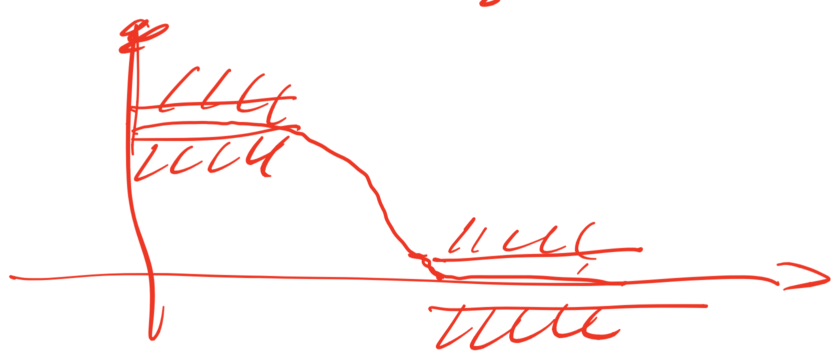
MEETING SPEC @ PASSBAND

$$\left(\frac{\Omega_p}{\Omega_c}\right)^6 = k_1 \; ; \; \Omega_c = \frac{\Omega_p}{k_1^{1/2N}} = \boxed{\Omega_c = \frac{.4716}{T}}$$

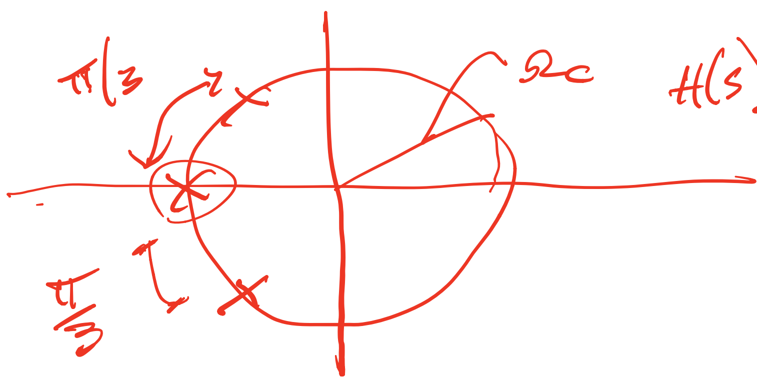
Ω_p ← $.15\pi/T$

MEETING SPECS @ STOPBAND

$$\left(\frac{\Omega_s}{\Omega_c}\right)^6 = k_2 \; ; \; \Omega_c = \frac{\Omega_s}{k_2^{1/2N}} = \frac{.5112}{T} = \Omega_c$$



BUTTERWORTH $N = 3$, $\Omega_c = .4716/T$



$$H(s) = \frac{\Omega_c^3}{(s+\Omega_c)\left(s-\Omega_c e^{j(\pi-\pi/3)}\right)\left(s-\Omega_c e^{j(\pi+\frac{\pi}{3})}\right)}$$

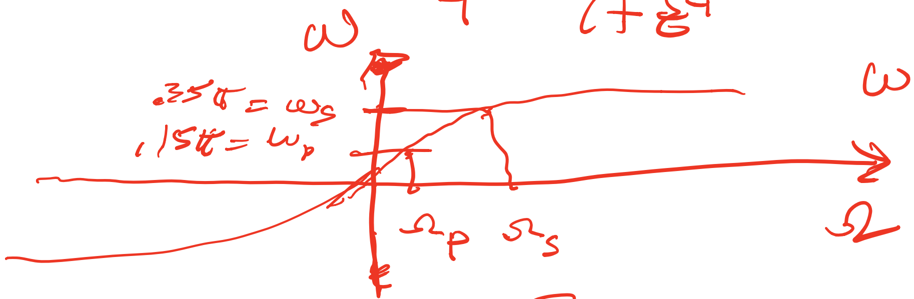
PARTIAL FRACTION EXPANSION

$$H(s) = \sum_{k=1}^{3} \frac{A_k}{s - s_k} \Rightarrow H(z) = \sum_{k=1}^{N} \frac{A_k}{1 - e^{s_k T} z^{-1}}$$

BILINEAR XFORM DESIGN

$$\rightarrow \quad s = \frac{2}{T} \frac{1 - z^{-1}}{1 + z^{-1}} \qquad \Omega = \frac{2}{T} \tan\left(\frac{\omega}{2}\right)$$

$$\omega = 2\tan^{-1}\left(\frac{\Omega T}{2}\right)$$



$$\text{NEED } \Omega_p = \frac{2}{T} \tan\left(\frac{.15\pi}{2}\right) = \frac{.4802}{T}$$

$$\frac{\Omega_s}{\Omega_p} > \frac{\omega_s}{\omega_p}$$

$$\Omega_s = \frac{2}{T} \tan\left(\frac{.35\pi}{2}\right) = \frac{1.226}{T}$$

$$N = \frac{1}{2} \frac{\log(k_2/k_2)}{\log(\Omega_s/\Omega_p)} = 2.4546$$

MATCH SPECS AT PASSBAND $\Rightarrow \Omega_c = \dfrac{\Omega_p}{k_1^{1/6}} = \dfrac{.4806}{T}$

$$H(s) = \frac{\Omega_c^3}{\left(s - \Omega_c e^{i\pi}\right)\left(s - \Omega_c e^{i\left(\pi - \frac{\pi}{3}\right)}\right)\left(s - \Omega_c e^{i\left(\pi + \frac{\pi}{3}\right)}\right)}$$

$$s = \frac{2}{T} \frac{1 - z^{-1}}{1 + z^{-1}}$$

BILINEAR

$$H(z) = \frac{.00854\left(1 + 3z^{-1} + 3z^{-2} + z^{-3}\right)}{1 - .2064z^{-1} + 1.5128z^{-2} - .386z^{-3}}$$

SEE THE PRINTED NOTES FOR THE COMPLETE IMPLEMENTATION!