Carnegie Mellon DSP Electrical & Computer ENGINEERING

Digital Signal Processing (18-491/18-691)
Spring Semester, 2023

IIR Filter Design Example

These notes summarize the design procedure for IIR filters as discussed in class on April 9.

Introduction:

We discuss in this lecture how to design a discrete-time lowpass filter using both the impulse invariance method and the bilinear transform method. The specs for the filter to be designed are

- Passband cutoff frequency: $\omega_p = 0.15\pi$
- Stopband cutoff frequency: $\omega_s = 0.35\pi$
- Passband ripple[1]: $-3 \text{ dB} \leq \left|H(e^{j\omega})\right| \leq 0 \text{ dB}$, $|\omega| \leq \omega_p$
- Stopband attenuation: $\left|H(e^{j\omega})\right| \leq -20 \text{ dB}$, $\omega_s \leq |\omega| \leq \pi$

In terms of the parameters δ_p and δ_s discussed in OSYP, this means that

$$1 - \delta_p = 10^{-3/20} = 0.7079$$

$$\delta_s = 10^{-20/20} = 0.1$$

For convenience we define two new parameters k_1 and k_2 that are closely related to δ_p and δ_s as follows:

$$k_1 \equiv \frac{1}{(1-\delta_p)^2} - 1 = 0.9953$$

$$k_2 \equiv \frac{1}{\delta_s^2} - 1 = 99$$

[1]Note that the values of passband and stopband attenuation are given in decibels (dB). If x has the dimensions of amplitude or magnitude (as opposed to energy or power), the value of x expressed in dB would be $20\log_{10}(x)$.

Finally, we are designing our filter by transforming the frequency response of a continuous-time Butterworth filter. As you will recall, the Butterworth filter has the transfer function

$$|H(j\Omega)|^2 = \frac{1}{1 + \left(\frac{\Omega}{\Omega_c}\right)^{2N}}$$

The parameter N specifies the number of poles in the prototype filter, and the parameter Ω_c specifies the critical cutoff frequency. Note that the magnitude response of the Butterworth filter decreases monotonically with increasing Ω, that the falloff in response as we transit from passband to stopband becomes sharper as N increases, and that $|H(j\Omega)|^2$ equals 1.0 when $\Omega = 0$ and 1/2 when $\Omega = \Omega_c$ for all values of N.

As you will recall from class, a Butterworth filter with parameters N and Ω_c has N poles that fall in the left half of the s-plane, in complex conjugate locations along a circle of radius Ω_c separated by angles of π/N.

Design procedure using impulse invariance

Determining design parameters

Using the impulse invariance design procedure, we have noted that the relation between frequency in the continuous-time and discrete-time domains is $\omega = \Omega T$, where $T$is merely a design parameter (and not necessarily the sampling period). Leaving T as an arbitrary constant for now, we obtain

$$\Omega_p = \omega_p / T = 0.47124/T$$

$$\Omega_s = \omega_s / T = 1.0996/T$$

At the passband edge frequency,

$$|H(j\Omega)|^2 = \frac{1}{1 + \left(\frac{\Omega_p}{\Omega_c}\right)^{2N}} = (1 - \delta_p)^2$$

With a few algebraic manipulations we obtain

$$\left(\frac{\Omega_p}{\Omega_c}\right)^{2N} = \frac{1}{(1 - \delta_p)^2} - 1 = k_1$$

Similarly, at the stopband edge frequency we have

$$|H(j\Omega)|^2 = \frac{1}{1 + \left(\frac{\Omega_s}{\Omega_c}\right)^{2N}} = \delta_s^2$$

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which produces

$$\left(\frac{\Omega_s}{\Omega_p}\right)^{2N} = \frac{1}{\delta_s^2} - 1 = k_2$$

Dividing, we obtain

$$\frac{(\Omega_p/\Omega_c)^{2N}}{(\Omega_s/\Omega_c)^{2N}} = \frac{k_1}{k_2} \quad \text{or}$$

$$N = \frac{1}{2}\frac{\log(k_2/k_1)}{\log(\Omega_s/\Omega_p)} = 2.7144$$

Note that in using impulse invariance,

$$\frac{\Omega_s}{\Omega_p} = \frac{\omega_s/T}{\omega_p/T} = \frac{\omega_s}{\omega_p}$$

so the design parameter T has no effect at all on the value of N that is needed. Furthermore, as expected, the value of N specified increases as either δ_p or δ_s decreases in magnitude (which implies decreasing the ripple in the passband and/or stopband) or as the ratio Ω_s/Ω_p decreases (which implies a decrease in the width of the transition band).

Since the number of poles must be an integer, we round up to $N = 3$.

Matching the frequency response exactly at passband produces

$$\left(\frac{\Omega_p}{\Omega_c}\right)^{2N} = k_1 \quad \text{or} \quad \Omega_c = \frac{\Omega_p}{k_1^{1/2N}} = \frac{0.4716}{T}$$

If we were instead to match the frequency response at stopband, we would obtain

$$\left(\frac{\Omega_s}{\Omega_c}\right)^{2N} = k_2 \quad \text{or} \quad \Omega_c = \frac{\Omega_s}{k_2^{1/2N}} = \frac{0.5112}{T}$$

In principle, any value of the critical frequency that satisfies $\frac{0.4716}{T} \leq \Omega_c \leq \frac{0.5112}{T}$ would be valid. As discussed in class, we will choose the value that meets the passband spec exactly, $\Omega_c = 0.4716/T$ because this value leaves the greatest margin for error at the stopband edge. Since the impulse invariance procedure always incurs a certain degree of aliasing, it will be expected that the *actual* response of the filter at the stopband edge will be larger than designed for. Hence, having the stopband edge response be overdesigned will leave more room for error to allow for the effects of aliasing.

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Prototype filter design

As noted above, the poles of a Butterworth filter lie in the left half of the s-plane on a circle of radius Ω_c in complex conjugate pairs separated by angles of π/N radians. Since there are three poles, this produces a filter with the system function

$$H(s) = \frac{\Omega_c^3}{(s - \Omega_c e^{j\pi})(s - \Omega_c e^{j(\pi+\pi/3)})(s - \Omega_c e^{j(\pi-\pi/3)})}$$

Plugging in the value of $\Omega_c = 0.4716/T$, we obtain (with the help of MATLAB)

$$H(s) = \frac{(0.4716/T)^3}{(s + 0.4716/T)(s - (-0.236 + 0.408j)/T)(s - (-0.236 - 0.408j)/T)}$$

Conversion to discrete-time form

Using partial fractions we can rewrite the system function of the continuous-time prototype filter as

$$H(s) = \sum_{k=1}^{N} \frac{A_k}{(s - s_k)}$$

where the parameters A_k are the continuous-time residues of the poles s_k.

Using the MATLAB routine `residue` we find the residues for the three poles, producing the transfer function

$$H(s) = \frac{0.4716}{s + 0.4716/T} + \frac{-0.2316 - 0.136j}{s - (-0.236 + 0.408j)/T} + \frac{-0.2316 + 0.136j}{s - (-0.236 - 0.408j)/T}$$

The corresponding discrete-time filter has the transfer function

$$H(z) = \sum_{k=1}^{N} \frac{A_k}{1 - e^{s_k T} z^{-1}}$$

Note that the poles of the continuous-time filter, s_k, are all of the form of a (generally complex) constant divided by the parameter T. Since s_k is multiplied by T wherever it appears in the equation for $H(z)$ above, the specific value chosen for T has no effect at all on the discrete-time filter that results from design process. Hence we frequently let $T = 1$ for simplicity. This produces the transfer function

$$H(z) = \frac{0.4716}{1 - e^{-0.4716} z^{-1}} + \frac{-0.2316 - 0.136j}{1 - e^{-0.236+0.408j} z^{-1}} + \frac{-0.2316 + 0.136j}{1 - e^{-0.236-0.408j} z^{-1}}$$

or

$$H(z) = \frac{0.4716}{1 - 0.624 z^{-1}} + \frac{-0.2316 - 0.136j}{1 - (0.7215 + 0.313j) z^{-1}} + \frac{-0.2316 + 0.136j}{1 - (0.7215 - 0.313j) z^{-1}}$$

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Combining the second and third terms we obtain

$$H(z) = \frac{0.4716}{1 - 0.624z^{-1}} + \frac{-0.472 + 0.341z^{-1}}{1 - 1.45z^{-1} + 0.624z^{-2}}$$

This expression is easily implemented using any of the techniques discussed in OSYP Chapter 6.

Design procedure using bilinear transformation

Determining design parameters

Here we will convert from continuous-time to discrete-time form using the bilinear transform

$$s = \frac{2}{T}\frac{1 - z^{-1}}{1 + z^{-1}}$$

This produces the nonlinear relationship between continuous-time frequency and discrete-time frequency

$$\Omega = \frac{2}{T}\tan(\omega/2)$$

Converting the critical frequencies and to their continuous-time counterparts produces

$$\Omega_p = \frac{2}{T}\tan(\omega_p/2) = 0.4802/T$$

$$\Omega_s = \frac{2}{T}\tan(\omega_s/2) = 1.226/T$$

We again use the design equation

$$N = \frac{1}{2}\frac{\log(k_2/k_1)}{\log(\Omega_s/\Omega_p)} = 2.4546$$

Note that although we still need to have 3 poles, the actual fractional value of N smaller because of the greater ratio of stopband to passband edge frequencies provided by the nonlinear frequency warping of the bilinear transform.

Matching the specs exactly at the passband edge frequency produces the value for the critical frequency of

$$\Omega_c = \frac{\Omega_p}{k_1^{1/2N}} = \frac{0.4806}{T}$$

Similarly, matching the specs exactly at the stopband edge frequency produces a value for the critical frequency of

$$\Omega_c = \frac{\Omega_s}{k_2^{1/2N}} = \frac{0.5698}{T}$$

Although in principle the critical frequency could be any value between the two frequencies above, we will match the specs exactly at the stopband edge frequency because that is what the `buttord` routine in MATLAB appears to do.

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Prototype filter design

As before, the transfer function for the Butterworth filter is

$$H(s) = \frac{\Omega_c^3}{(s - \Omega_c e^{j\pi})(s - \Omega_c e^{j(\pi+\pi/3)})(s - \Omega_c e^{j(\pi-\pi/3)})}$$

Using the MATLAB routine `poly` and the value for that we had obtained we find that the transfer function of the prototype filter is

$$H(s) = \frac{0.1850/T^3}{s^3 + \frac{1.1397 s^2}{T} + \frac{0.6494 s}{T^2} + \frac{0.1850}{T^3}}$$

Conversion to discrete-time form

We can now substitute directly the bilinear transform relationship

$$s = \left(\frac{2}{T}\frac{1 - z^{-1}}{1 + z^{-1}}\right)$$

This produces

$$H(s) = \frac{0.1850/T^3}{\left(\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}\right)^3 + \frac{1.1397\left(\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}\right)^2}{T} + \frac{0.6494\left(\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}\right)}{T^2} + \frac{0.1850}{T^3}}$$

Note that the design parameter T appears in cubic form in the denominator of every term in numerator and denominator. Hence its specific value does not matter and we can cancel it from the design equation. The value of the parameter T is frequently set to equal 2 to simplify the expression for the bilinear transform.

We also note that simplifying the expression above is an algebraic nightmare. Using the MATLAB routine `bilinear` with the poles and (null) zeros of $H(s)$ as input parameters (or the outputs of the MATLAB `butter` routine), and a value of 1 for the parameter T, we obtain

$$H(z) = \frac{0.0132(1 + 3z^{-1} + 3z^{-2} + z^{-3})}{1 - 1.9017z^{-1} + 1.3315z^{-2} - 0.3244z^{-3}}$$

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MATLAB implementation of the bilinear transformation approach

The following MATLAB commands implement the results above, assuming a value of 1 for the parameter T in these calculations:

```
wp = .15*pi;
ws = .35*pi;
Wp = 2*tan(wp/2);
Ws = 2*tan(ws/2);
```

Getting the parameters for the continuous-time Butterworth filter. (MATLAB matches the specifications at the stopband edge, as noted above.)
`[N,Wn] = buttord(Wp,Ws,3,20,'s')`

Calculating the locations of the pole locations on the Butterworth circle (assumes $N = 3$):
`poles = Wn*[-1 exp(j*(pi-(pi/3))) exp(-j*(pi-(pi/3)))]`

Forming the denominator and numerator polynomials for the continuous-time transfer function:

```
as = poly(poles)
bs = [0 0 0 Wn^3]
```

Checking the continuous-time frequency response:
`freqs(bs,as)`

In the above sequence of commands we found the transfer function for $H(s)$ by hand. A faster and easier way to get the same results is to use the MATLAB `butter` routine to get the same result:
`[bs,as] = butter(N,Wn,'s')`
Checking the frequency response:
`freqs(bs,as)`

Now invoking the bilinear transform using the MATLAB `bilinear` routine:
`[b,a] = bilinear(bs,as,1)`

Note that I used a value of 1 for the parameter `Fs` which represents sampling frequency (the reciprocal of the OSYP parameter T).

Checking the discrete-time frequency response:

```
close all
[h,w] = freqz(b,a);
plot((w/pi),20*log10(abs(h)))
axis([0,1,-30,10])
```

By mousing over the figure produced by these steps you will notice that when the x-axis is equal to .15 (corresponding to a discrete-time frequency of $.15\pi$), the y-axis is at -20 (corresponding to a gain of -20 dB), as specified.

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