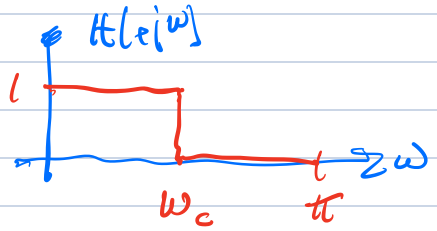
3/25/24

INTRO TO DIGITAL FILTER DESIGN

(DTSP 7.0–7.3)

IDEAL FILTERS



CTFT

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Omega)\, e^{j\Omega t}\, d\Omega$$

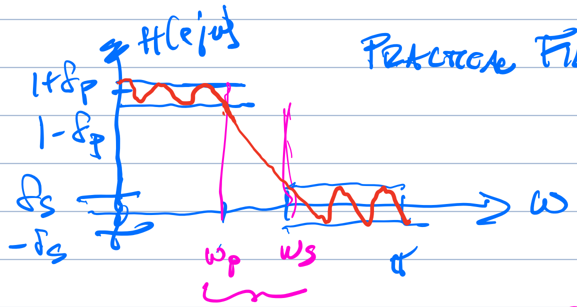
$$X(j\Omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\Omega t}\, dt$$

DTFT

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

PRACTICAL FILTER

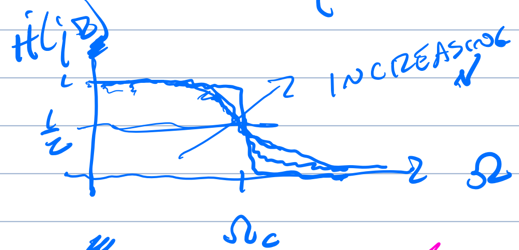


$\Delta\omega = \omega_s - \omega_p$ TRANSITION BANDWIDTH

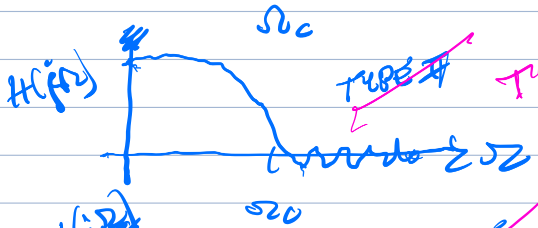
GOAL: FOR IIR DESIGN CONVERT A "GOOD" CT IIR DESIGN INTO A "GOOD" DT DESIGN

1. WHAT IS "GOOD"?
2. HOW TO DO CONVERSION

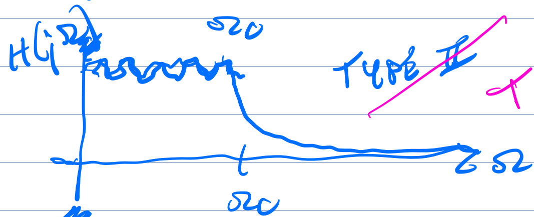
EXAMPLES of "GOOD" CT DESIGNS...



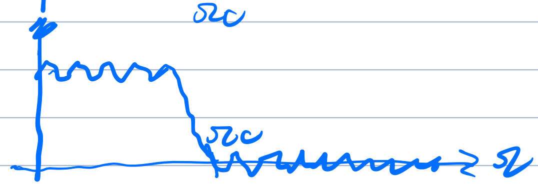
BUTTERWORTH



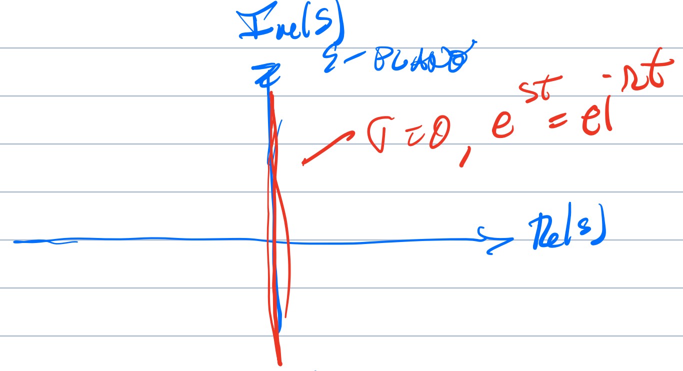
CHEBYSCHEV



ELLIPTIC FILTER



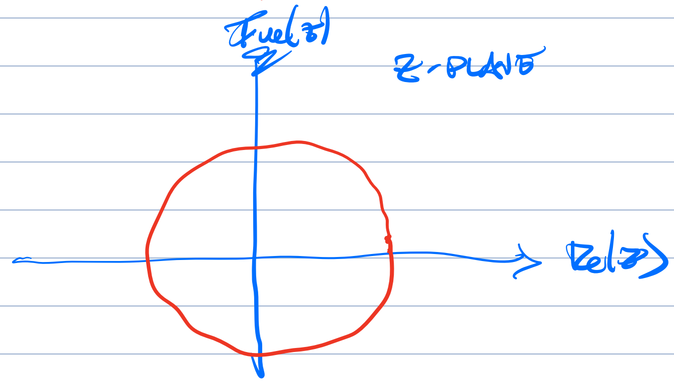
LAPLACE v. Z-TRANSFORMS



e^{st}, $s = \sigma + j\Omega$

$e^{(\sigma + j\Omega)t} = e^{\sigma t} e^{j\Omega t}$

S-PLANE		Z-PLANE
$j\Omega$-AXIS	$\Longleftrightarrow$	$\lvert z\rvert = 1$
LEFT HALF of S-PLANE	$\Longleftrightarrow$	$\lvert z\rvert < 1$
RIGHT HALF of S-PLANE	$\Longleftrightarrow$	$\lvert z\rvert > 1$



POSSIBLE MAPPINGS of S-PLANE TO Z-PLANE

$\boxed{h_c(t) \longrightarrow h[n]}$ IMPULSE INVARIANCE

★ $s_c(t) \longrightarrow s[n]$ STEP INVARIANCE

★ DIFF. EQ. $\longrightarrow$ DIFF. EQ. DIFF EQS

$\boxed{H_c(j\Omega) \longrightarrow H(e^{j\omega})}$ BILINEAR TRANSFORMATION

QUESTIONS ABOUT MAPPINGS:

★ STABLE IN CT $\longrightarrow$ STABLE IN DT?

★ FREE RESP IN CT $\longrightarrow$ FREE RESP IN DT?

MAPPING FROM CT TO DT, ...

METHOD I IMPULSE INVARIANCE

Let $h[n] = T_d \, h_c(t)\big|_{t=nT}$

Consider $h_c(t) = e^{s_k t} u(t)$

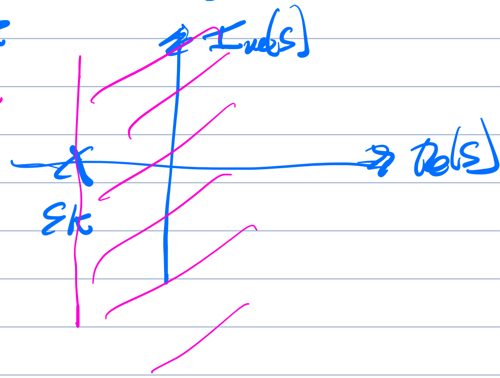
$$H(s) = \int_{-\infty}^{\infty} h_c(t) e^{-st} \, dt = \int_{-\infty}^{\infty} e^{s_k t} u(t) e^{-st} \, dt$$

$$= \int_{0}^{\infty} e^{t(s_k - s)} \, dt = \frac{1}{s_k - s} \left[e^{t(s_k - s)} \right] \Big|_{t=0}^{\infty}$$

$Re[s_k - s] < 0$

$\int e^{at} \, dt = \frac{1}{a} e^{at}$

$$= \frac{1}{s_k - s} [0 - 1] = \frac{1}{s - s_k}, \quad Re[s_k - s] < 0$$



$$h[n] = T h_c(t)\big|_{t=nT} = T e^{s_k t}\big|_{t=nT}$$

$$= T e^{s_k n T} = T \left(e^{s_k T} \right)^n u[n]$$

$$H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n} = T \sum_{n=0}^{\infty} \left(e^{s_k T} z^{-1} \right)^n = \frac{T}{1 - e^{s_k T} z^{-1}}$$

STABLE IN S-PLANE IF $Re(s_k) < 0$

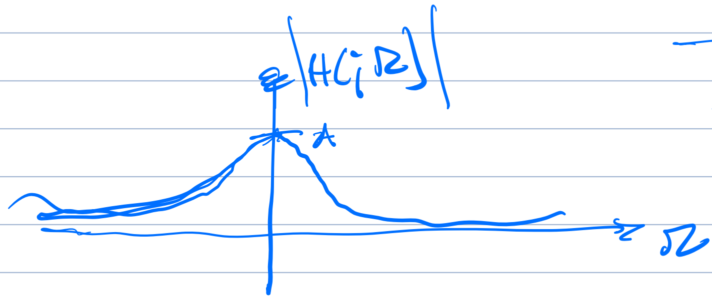
IN Z-PLANE $z_k = e^{s_k T} = e^{(\sigma + j\Omega)T} = e^{\sigma T} \cdot e^{j\Omega T}$

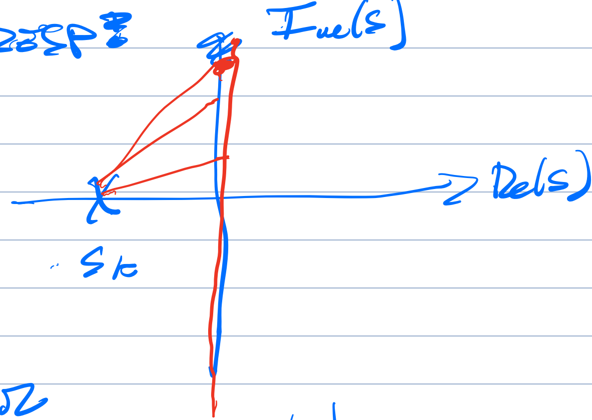
IF $\sigma < 0 \Rightarrow |z_k| < 1$

STABLE IN S $\Rightarrow$ STABLE IN Z

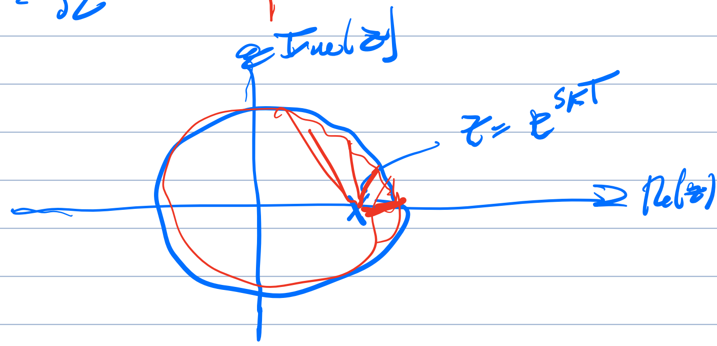
WHAT ABOUT FREQ RESP?

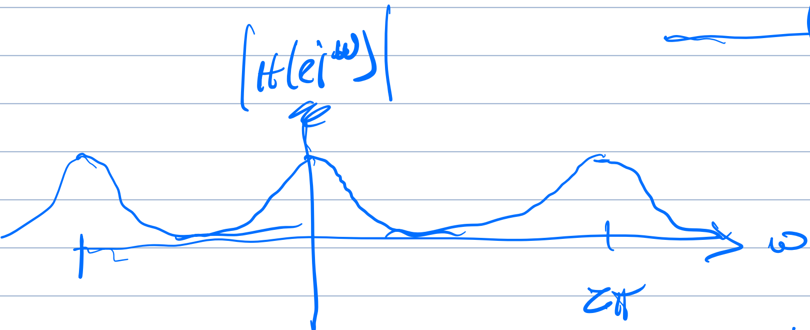
IN S-PLANE



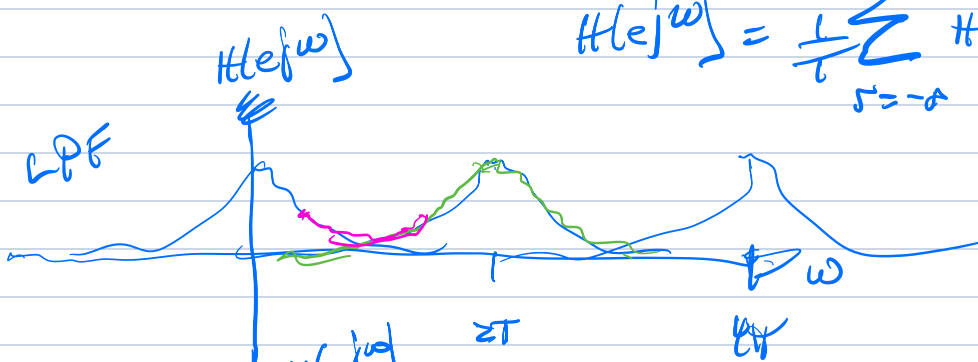


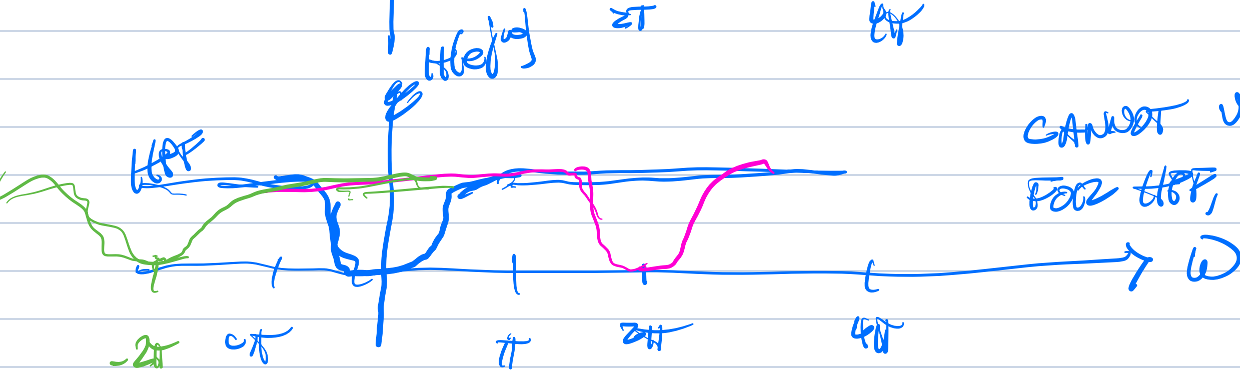
IN DT, Z-PLANE



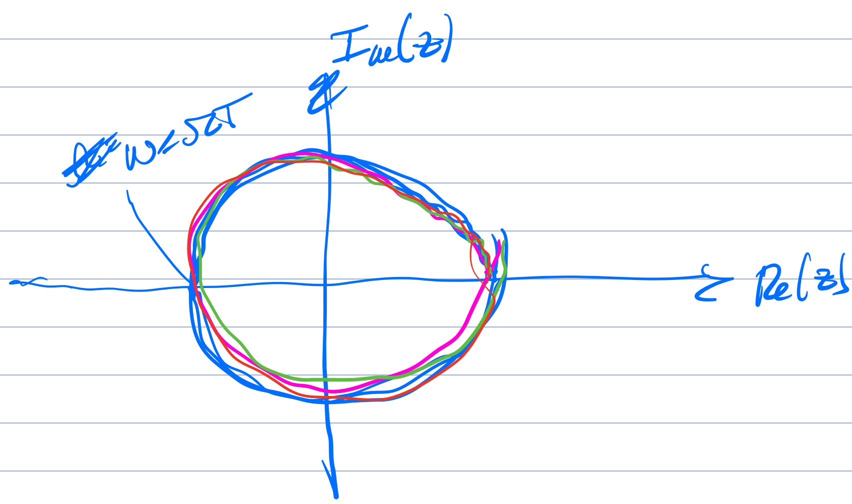


$$H(e^{j\omega}) = \frac{1}{T}\sum_{r=-\infty}^{\infty} H\left(j\left(\Omega - \frac{2\pi r}{T}\right)\right)$$





CANNOT USE FOR HPF, NOTCH FILTERS ETC.



$\omega = \Omega T$

CAN WE MAP $j\Omega$-AXIS OF S-PLANE INTO UNIT CIRCLE OF Z-PLANE ONCE ??

$\boxed{\text{THE BILINEAR TRANSFORM}}$

CONSIDER
$$s = \frac{2}{T}\,\frac{1-z^{-1}}{1+z^{-1}}$$

$$s = \frac{2}{T}\,\frac{z-1}{z+1}$$

$$sTz + sT = 2z - 2$$

$$z(sT-2) = -sT-2$$

$$z = \frac{-sT-2}{\cancel{s}T-\cancel{2}2} = \frac{2+sT}{2-sT} = \frac{1+\frac{sT}{2}}{1-\frac{sT}{2}}$$

Let $s = j\Omega \implies z = \dfrac{1+j\frac{\Omega T}{2}}{1-j\frac{\Omega T}{2}}$

$$|z|^2 = \frac{\left|1+j\frac{\Omega T}{2}\right|^2}{\left|1-j\Omega T\right|^2} = \frac{1+\left|\frac{\Omega T}{2}\right|^2}{1+\left|\frac{\Omega T}{2}\right|^2}$$

$j\Omega$-AXIS IN S MAPS TO UNIT CIRCLE IN Z

Now let $s = \sigma + j\Omega$

$$z = \frac{1 + \frac{sT}{2}}{1 - \frac{sT}{2}} = \frac{1 + \frac{(\sigma + j\Omega)T}{2}}{1 - \frac{(\sigma + j\Omega)T}{2}} = \frac{\frac{1+\sigma T}{2} + j\frac{\Omega T}{2}}{1 - \frac{\sigma T}{2} - j\frac{\Omega T}{2}}$$

$$|z|^2 = \frac{\left(\frac{1+\sigma T}{2}\right)^2 + \left(\frac{\Omega T}{2}\right)^2}{\left(1 - \frac{\sigma T}{2}\right)^2 + \left(\frac{\Omega T}{2}\right)^2}$$

LHP S-PLANE

$\sigma < 0$

$\Rightarrow$ INSIDE of U.C.

HOW DOES $j\Omega$-AXIS MAP INTO UNIT CIRCLE?

$$s = \frac{2}{T}\,\frac{1 - z^{-1}}{1 + z^{-1}} \qquad s = j\Omega$$

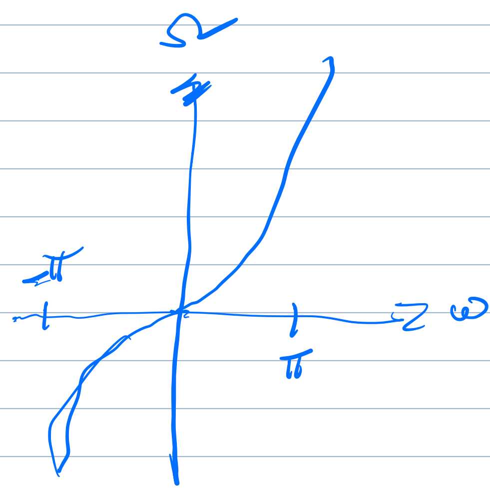
$$z = e^{j\omega}$$

$$j\Omega = \frac{2}{T}\,\frac{1 - e^{-j\omega}}{1 + e^{-j\omega}}$$

$$j\Omega = \frac{2}{T}\,\frac{\cancel{e^{-j\frac{\omega}{2}}}\left(e^{+j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}\right)}{\cancel{e^{-j\frac{\omega}{2}}}\left(e^{j\frac{\omega}{2}} + e^{-j\frac{\omega}{2}}\right)} =$$

$$\cancel{j}\Omega = \frac{2}{T}\,\frac{2\cancel{j}\sin\left(\frac{\omega}{2}\right)}{2\cos\left(\frac{\omega}{2}\right)}$$

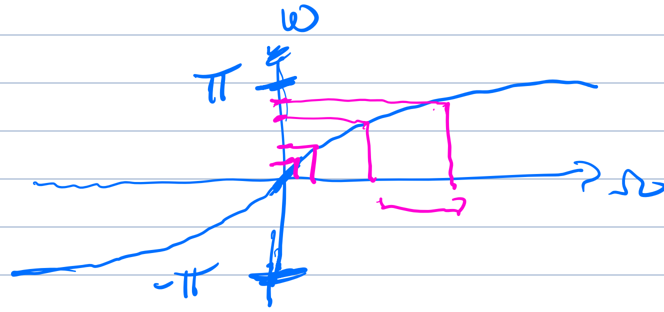
$$\boxed{\Omega = \frac{2}{T}\tan\left(\frac{\omega}{2}\right)}$$



$$\frac{\Omega T}{2} = \tan\left(\frac{\omega}{2}\right)$$

$$\tan^{-1}\left(\frac{\Omega T}{2}\right) = \frac{\omega}{2}$$

$$\boxed{\omega = 2\tan^{-1}\left(\frac{\Omega T}{2}\right)}$$



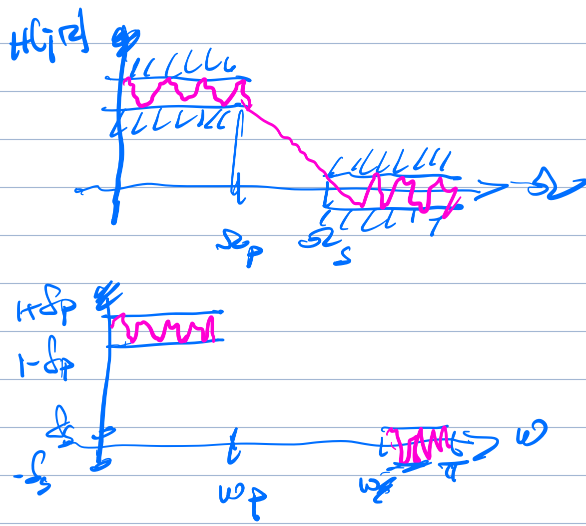
x SMALL $\sin(x) \approx x$ $\tan(x) \approx x$

$\cos(x) \approx 1$ $\tan^{-1}(x) \approx x$

FOR $|\omega|$ SMALL $\omega \approx 2\left(\frac{\Omega T}{2}\right)$

$\omega \approx \Omega T$

APPLICATION TO FILTER DESIGN



$H(j\Omega) \rightarrow H(e^{j\omega})$

USING BLT

$$\omega = 2\tan^{-1}\left(\frac{\Omega T}{2}\right)$$

$$\Omega = \frac{2}{T}\tan\left(\frac{\omega}{2}\right)$$