3/11/24

INTRODUCTION TO THE FFT (OS&P 9.0, 9.2)

DFT: $$x[n] = \frac{1}{N}\sum_{k=1}^{N-1} X[k]\, e^{j\frac{2\pi n k}{N}}$$

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi n k}{N}}$$

$W_N = e^{-j\frac{2\pi}{N}}$ "TWIDDLE FACTORS"

$$x[n] = \frac{1}{N}\sum_{k=1}^{N-1} X[k]\, W_N^{-nk}$$

$$X[k] = \sum_{n=0}^{N-1} x[n]\, W_N^{nk}$$

COMPUTATION $= N^2$

MEASURES OF COMPUTATIONAL COMPLEXITY

- ADDITIONS
- \# MULTIPLICATIONS
- DIVISIONS
- MEMORY USAGE
- SCALABILITY

COST OF LINEAR FILTERING

IF $x[n]$ IS N POINTS LONG
FINITE $h[n]$ IS M POINTS LONG

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

\# of MULTS $\approx MN$

IF $M=N$, MULTS $\approx O(N^2)$

USING DFTS ....

1. $M+N-1$ PT. DFT of $x[n] \rightarrow (M+N-1)^2$ MULTS.
2. $M+N-1$ PT. DFT of $h[n] \rightarrow (M+N-1)^2$ MULTS
3. PRODUCE $X[k] \cdot H[k] = Y[k] \rightarrow M+N-1$
4. $M+N-1$ PT. IDFT of $Y[k] \rightarrow (M+N-1)^2$ MULTS

TOTAL MULTS $= 3(M+N-1)^2 + N+M-1$

FOR $N \approx M \Rightarrow 3N^2+N \to \approx 3N^2$ MULTS, LARGE N

$O(N^2)$

COOLEY-TUKEY DECIMATION IN TIME

$$X[k] = \sum_{n=0}^{N-1} x[n]\, W_N^{nk} = \sum_{n=0}^{N-1} x[n]\, e^{-j2\pi nk/N}, \quad W_N^{nk} = e^{-j2\pi nk/N}$$

$$X[k] = \sum_{\substack{n=0 \\ n \text{ EVEN}}}^{N-1} x[n]\, W_N^{nk} + \sum_{\substack{n=0 \\ n \text{ ODD}}}^{N-1} x[n]\, W_N^{nk}$$

Let $n = 2r$ FOR n EVEN, $n = 2r+1$, n ODD

$$X[k] = \sum_{r=0}^{\frac{N}{2}-1} x[2r]\, W_N^{2rk} + \sum_{r=0}^{\frac{N}{2}-1} x[2r+1]\, W_N^{(2r+1)k}$$

$$W_N^{2rk} = e^{-j\frac{2\pi}{N}\cdot 2rk} = e^{-j\frac{2\pi}{N/2} rk} = W_{\frac{N}{2}}^{rk}$$

$$W_N^{(2r+1)k} = W_N^{k}\, W_N^{2rk} = W_N^{k}\, W_{\frac{N}{2}}^{rk}$$

$$X[k] = \underbrace{\sum_{r=0}^{\frac{N}{2}-1} x[2r]\, W_{\frac{N}{2}}^{rk}}_{G[k] = \frac{N}{2}\text{-PT. DFT of EVEN SAMPLES}} + W_N^{k} \underbrace{\sum_{r=0}^{\frac{N}{2}-1} x[2r+1]\, W_{\frac{N}{2}}^{rk}}_{H[k] = \frac{N}{2}\text{-PT. DFT of ODD SAMPLES}}$$

IS THIS USEFUL?

CONSIDER $N=8$ ORIGINAL DFT $\Rightarrow 8^2 = 64$ MULTS

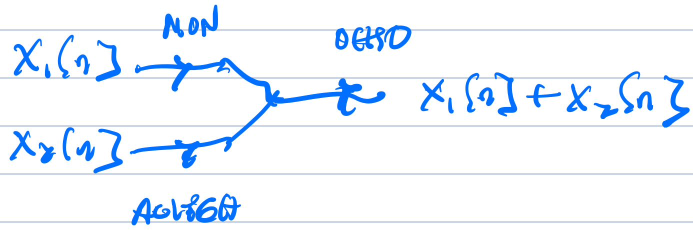
REVISED DFT COMP $\Rightarrow (2)(4^2) = 32$ MULTS

$+8$

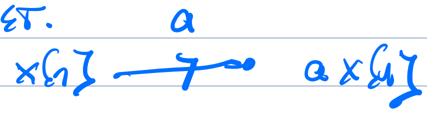
$= 4$ MULTS

SIGNAL FLOW GRAPH NOTATION

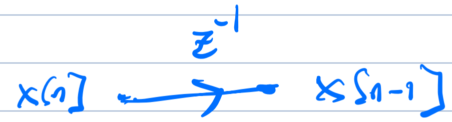
ADDITION



MULT BY CONST.



UNIT DELAY



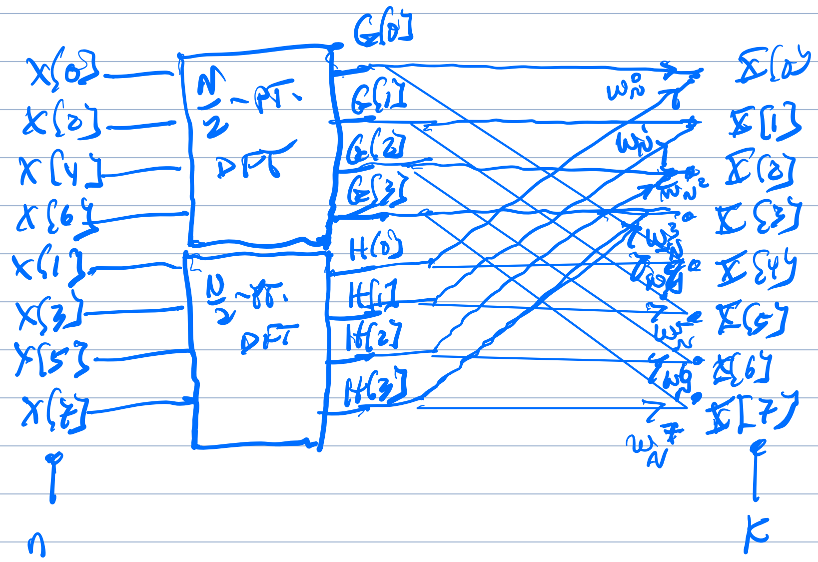
SIGNAL FLOW REP of N-PT. DFT

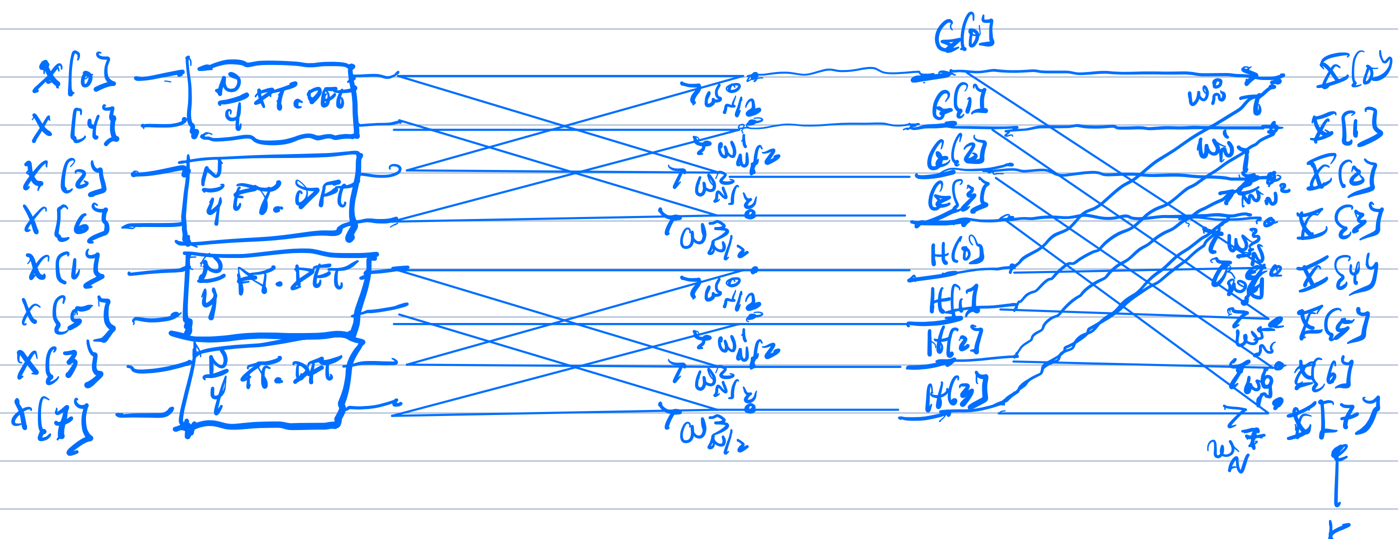
$$X[k] = G[k] + W_N^k H[k]$$

$G[k]$: $\frac{N}{2}$ PT. DFT of EVENS

$H[k]$: $\frac{N}{2}$ PT. DFT of ODDS

$N=8$



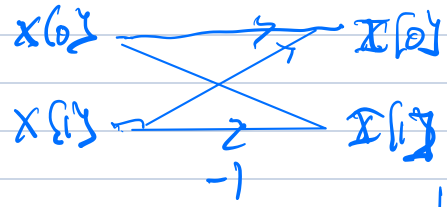


WHAT IS A 2-PT. DFT?

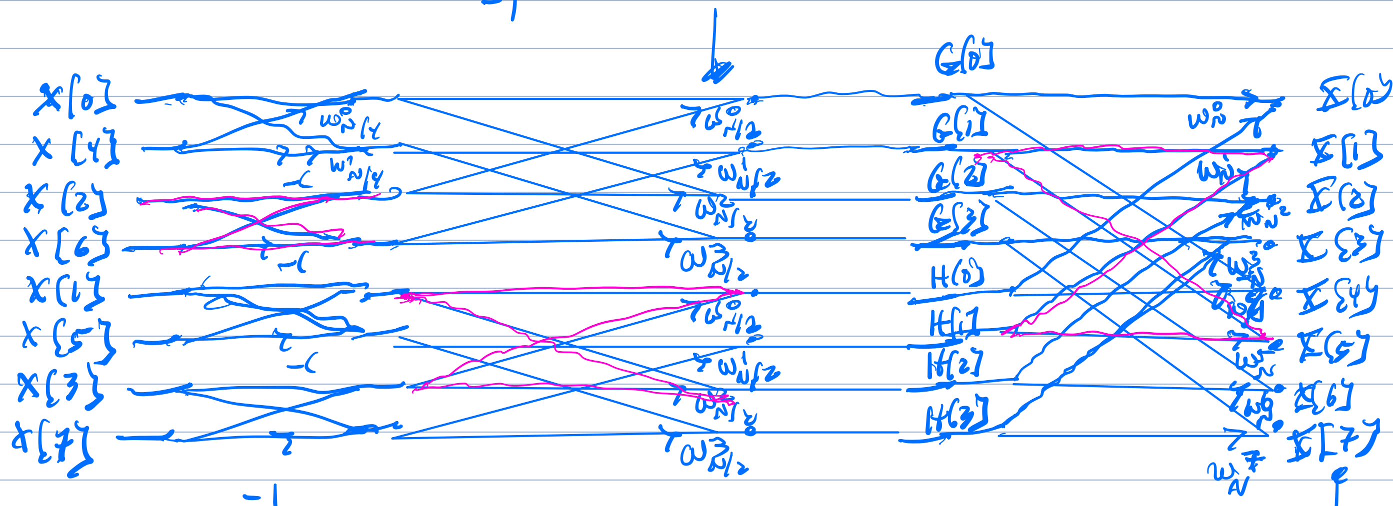
$$X[k] = \sum_{n=0}^{1} x[n]\, e^{-j2\pi nk/2}$$

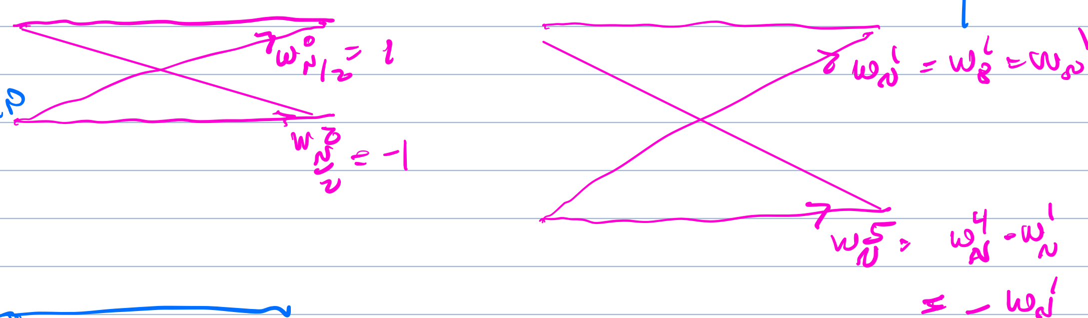
$$X[0] = x[0] + x[1]$$

$$X[1] = [0] - x[1]$$



"BASIC BUTTERFLY"





TOTAL # MULTS

= TOTAL # COLUMNS · TOTAL # MULTS / COLUMN

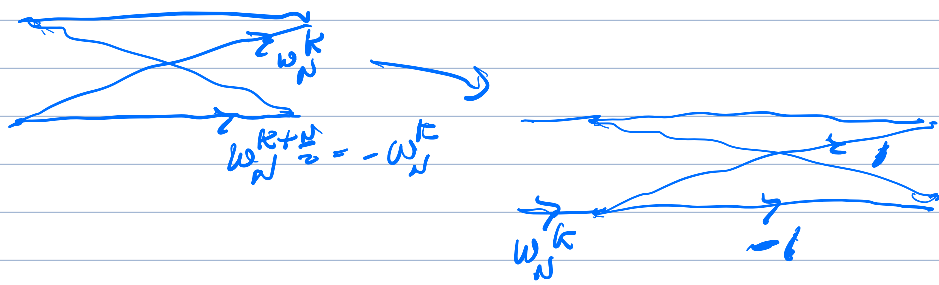
$N = 8$ $= (3)(8)$

$= N \log_2 N$

MULTS $= N \log_2 N$

$O(N \log N)$

REAL MULTS IS $\frac{N}{2} \log_2 N$



$x[0]$	0 0 0	0
$x[4]$	1 0 0	1
$x[2]$	0 1 0	2
$x[6]$	1 1 0	3
$x[1]$	0 0 1	4
$x[5]$	1 0 1	5
$x[3]$	0 1 1	6
$x[7]$	1 1 1	7

"BIT-REVERSED ORDER"

COMPUTATIONAL SAVINGS:

Let $\alpha_1 = \dfrac{N \log_2 N}{N^2}$, $\alpha_2 = \dfrac{3 \log_2 N + N}{N^2}$

N	α_1	α_2
16	.25	.813
32	.156	.5
64	.0935	.297
128	.055	.171
256	.031	.097
1024	.0097	.080