Carnegie Mellon DSP Electrical & Computer ENGINEERING

Fundamentals of Signal Processing (18-491)
Spring Semester, 2019

INTRODUCTION TO THE DFS AND THE DFT

Notes: This brief handout contains in very brief outline form the lecture notes used for a video lecture in a previous year introducing the DFS and the DFT. This material is covered in Secs. 8.0 through 8.6 in the text. This material is covered over several lectures this year, and there may be some deviations from the presentation in OSYP.

I. Introduction

In previous lectures we discussed the relationship between pole and zero locations in the z-plane of an LSI system and the magnitude and phase of the transfer function of that system. We also introduced the idea of the discrete Fourier transform (DFT). The DFT is important because it is the mathematical relation that is implemented by the various Fast Fourier Transform (FFT) algorithms. In today's lecture we will discuss

- relationships between periodic and finite-duration time functions
- the discrete Fourier series (DFS) for periodic time functions
- the discrete Fourier transform (DFT) for finite-duration time functions
- circular versus linear convolution

The discrete Fourier series (DFS) is used to represent periodic time functions and the DFT is used to represent finite-duration time functions. The two representations are virtually identical mathematically, and they are closely related because a finite-duration time function can be thought of as a single period of a periodic time function. Conversely, a periodic time function can be easily constructed from a finite-duration time function simply by repeating the finite-duration sequence over an over again, *ad infinitum.*

We adopt the notation used by OSYP to distinguish these functions: let $\tilde{x}[n]$ represent a periodic discrete-time sequence with period N, and let $x[n]$ (without the tilde) represent a finite-duration sequence that is nonzero for $0 \le n \le N-1$. We than note (trivially) that

$$x[n] = \tilde{x}[n], 0 \le n \le N-1 \text{ and}$$

$$\tilde{x}[n] = x(n \bmod N) \equiv x[((n))_N]$$

II. The Discrete Fourier Series (DFS)

A. Definitions of the DFS

If a time function $\tilde{x}[n]$ is periodic has period *N,* we can write

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]e^{jk\frac{2\pi}{N}n} \text{ and}$$

$$\tilde{X}[k] = \sum_{n=0}^{N-1}\tilde{x}[n]e^{-jk\frac{2\pi}{N}n}$$

There are only *N* unique frequency components because

$$e^{j2\pi kn/N} = e^{j2\pi(k+rN)n/N} = e^{j2\pi kn/N}e^{j2\pi rnN/N} = e^{j2\pi kn/N} \quad (1)$$

for integer *k*, *n*, and *r*.

Comments:

1. Both the time function $\tilde{x}[n]$ and the Fourier series coefficients $\tilde{X}[k]$ are periodic with period *N,* so they are represented by only *N* distinct (possibly complex) numbers.

2. The series can be evaluated over any consecutive of values of *n* or *k* of length *N*.

3. The frequency components of the DFS are the complex exponentials $e^{j\frac{2\pi}{N}kn}$, where the frequency $\frac{2\pi}{N}$ can be considered to be the "fundamental frequency" of the periodic waveform and all other frequencies are integer multiples of it.

4. The frequency components are *N* equally-spaced samples of the frequencies of the DTFT, or alternatively they represent *N* equally-spaced locations around the unit circle of the *z*-plane.

We commonly use the notational shorthand $W_N = e^{-j2\pi/N}$, so the DFS equations can be rewritten as

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]W_N^{-nk} \text{ and}$$

$$\tilde{X}[k] = \sum_{n=0}^{N-1}\tilde{x}[n]W_N^{nk}$$

B. The DTFT of periodic time functions

As we had discussed previously, we can apply the properties of ordinary DTFTs to express the DTFT of any periodic time function in terms of its DFS coefficients. If $\tilde{x}[n]$ has period N and DFS coefficients $\tilde{X}[k]$, the DTFT of $\tilde{x}[n]$ can be written as

$$\tilde{X}(e^{j\omega}) = \sum_{r=-\infty}^{\infty} \sum_{k=0}^{N-1} \frac{2\pi}{N}\tilde{X}[k]\delta\left(\omega - \frac{2\pi k}{N} - 2\pi r\right) = \sum_{k=-\infty}^{\infty} \frac{2\pi}{N}\tilde{X}[k]\delta\left(\omega - \frac{2\pi k}{N}\right)$$

In other words, the DTFT of a periodic time function $\tilde{x}[n]$ with period N consists of a sequence of delta functions located at frequencies $\omega = 2\pi k/N$. The areas of these delta functions are all $2\pi/N$ times the corresponding DFS coefficients.

C. Properties of the DFS

In general, the properties of the DFS are very similar to what we would expect from the DTFT, except that the functions considered are periodic. We only touch on a few important properties in the lecture, but a more complete list is available in OSYP Table 8.1.

1. **Linearity**: $a\tilde{x}_1[n] + b\tilde{x}_2[n] \Leftrightarrow a\tilde{X}_1[n] + b\tilde{X}_2[n]$ provided that the period of the two time functions is the same.

2. **Time shift:** $\tilde{x}[n-m] \Leftrightarrow W_N^{km}\tilde{X}[k]$

3. **Multiplication by a complex exponential:** $\tilde{x}[n]W_N^{-rn} \Leftrightarrow \tilde{X}[k-r]$

Note that in the previous two properties, shift in one domain corresponds to multiplication in the other by a complex exponential, both of which may be easier to evaluate when you apply the definition for W_N.

4. Multiplication-convolution: $\tilde{x}_1[n]\tilde{x}_2[n] \Leftrightarrow \frac{1}{N}\sum_{r=0}^{N-1} \tilde{X}_1[r]\tilde{X}_2[k-r]$

This relationship was derived in class in the opposite domain. The sum on the right hand side is very important for our work and is referred to as *periodic convolution* or *circular convolution.* The operator for circular convolution is normally written as an asterisk with a circle around it, possibly with an accompanying number that indicates the size of the convolution (which matters).

III. The Discrete Fourier Transform (DFT)

A. Introduction

The DFT is used to represent the frequency components of a finite-duration function. In reality it is nothing more than the DFS for the corresponding periodic time functions, imagining that we have only a single period of that function. It is always important to keep in mind the implied periodicity of the functions in both time and frequency domains to understand the properties of the transform.

Specifically, the DFT relationships are written as:

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{jk\frac{2\pi}{N}n} = \frac{1}{N}\sum_{k=0}^{N-1} X[k]W_N^{-kn} \text{ and}$$

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-jk\frac{2\pi}{N}n} = \sum_{n=0}^{N-1} x[n]W_N^{kn}$$

Comments:

1. While strictly speaking the DFT and its inverse are zero for values of n and k that do not lie between 0 and $N-1$, in reality we know that both of these functions are implicitly periodic.
2. The DFT can be thought of as the representation that is obtained by sampling the DTFT of a finite-duration time function at $\omega = 2\pi k/N$, which is to say N evenly spaced samples along the ω axis starting at $\omega = 0$, or equivalently, the DFT coefficients can be thought of as samples of the z-transform of the time function at locations $z = e^{j2\pi k/N}$ in the z-plane, or N evenly-spaced locations along the unit circle of the z-plane beginning at $z = 1$. Mathematically we can write

$$X[k] = X(e^{j\omega]})\Big|_{\omega = 2\pi k/N} = X(z)\Big|_{z = e^{j2\pi k/N}}$$

B. Properties of the DFT

The properties of the DFT are virtually identical to the corresponding DFS relationships, except that we are talking about finite-duration rather than periodic time functions. A much more complete list may be found in OSYP Table 8.2.

1. **Linearity**: $ax_1[n] + bx_2[n] \Leftrightarrow aX_1[n] + bX_2[n]$ provided that the two DFTs are of equal size

2. **Time shift:** $x[((n-m))_N] \Leftrightarrow W_N^{km}X[k]$

3. **Multiplication by a complex exponential:** $x[n]W_N^{-rn} \Leftrightarrow X[((k-r))_N]$

Note the use of the modulo-N notation to denote *circular shift*. Again, this arises because both time and frequency functions are implicitly periodic.

4. **Multiplication in time:** $x_1[n]x_2[n] \Leftrightarrow \frac{1}{N}\sum_{r=0}^{N-1} X_1[r]X_2[((k-r))_N]$

5. Convolution in time: $\sum_{m=0}^{N-1} x_1[m]x_2[((n-m))_N] = X_1[k]X_2[k]$

Note that the previous equation means that if we take *N*-point DFTs of two time functions $x_1[n]$ and $x_2[n]$, multiply the two sets of DFT coefficients together, and compute the inverse *N*-point DFT of the product, the resulting sequence is the *N*-point *circular* convolution of the original time sequences.

There are also a number of symmetry properties. They follow the corresponding properties for DTFTs with the understanding that $X[-k] = X[((-k))_N] = X[N-k]$.

C. Circular versus linear convolution

As in the case of DTFTs, we frequently find it advantageous to perform convolution using transform techniques, specifically by computing the Fourier transforms of the functions to be convolved, multiplying them together, and computing the corresponding inverse transforms. The problem is that with we normally desire the result that we would have had with linear convolution of these time functions, and using DFTs we really obtain the corresponding circular convolutions.

One useful way of thinking about circular convolution is that the circular convolution of two finite-duration time functions can be considered equivalent to the linear convolution of the same two functions followed by *N*-point aliasing, where *N* is the size of the circular convolution.

We showed in class that if we perform the circular convolution of two finite-duration functions of lengths N_1 and N_2 by multiplying together their *N*-point DFTs and computing the inverse *N*-point DFT, that result will equal the corresponding linear convolution if $N \geq N_1 + N_2 - 1$.