THE DFS AND THE DFT

(OSYP 8.3-8.6)

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$$

DTFT

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

DTFS | DFS

$$\tilde{x}[n] = \tilde{x}[n-N]$$

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}[k]\, e^{j\frac{2\pi nk}{N}}$$

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n]\, e^{-j\frac{2\pi nk}{N}}$$

Also: $\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}[k]\, e^{jk\omega_0 n}$

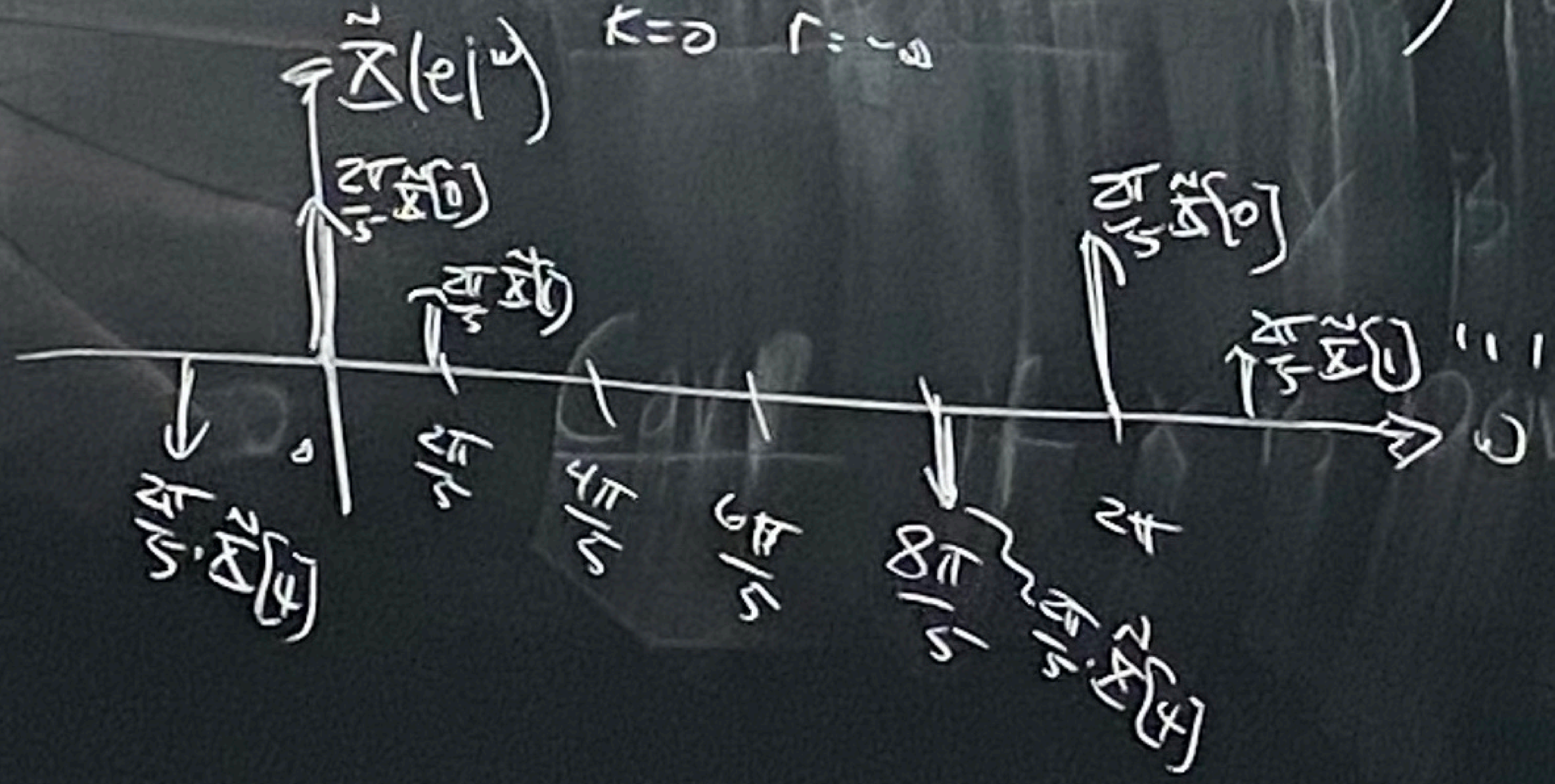
$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n]\, e^{-jk\omega_0 n}, \quad \omega_0 = \frac{2\pi}{N}$$

THE DTFT of PERIODIC TIME FUNCTIONS

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}[k]\, e^{jk\omega_0 n} \quad ; \quad \omega_0 = \frac{2\pi}{N}$$

$$\frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}[k]\, e^{jk\omega_0 n} \Longleftrightarrow \frac{2\pi}{N}\sum_{k=0}^{N-1} \tilde{X}[k] \sum_{r=-\infty}^{\infty} \delta(\omega - k\omega_0 - 2\pi r)$$

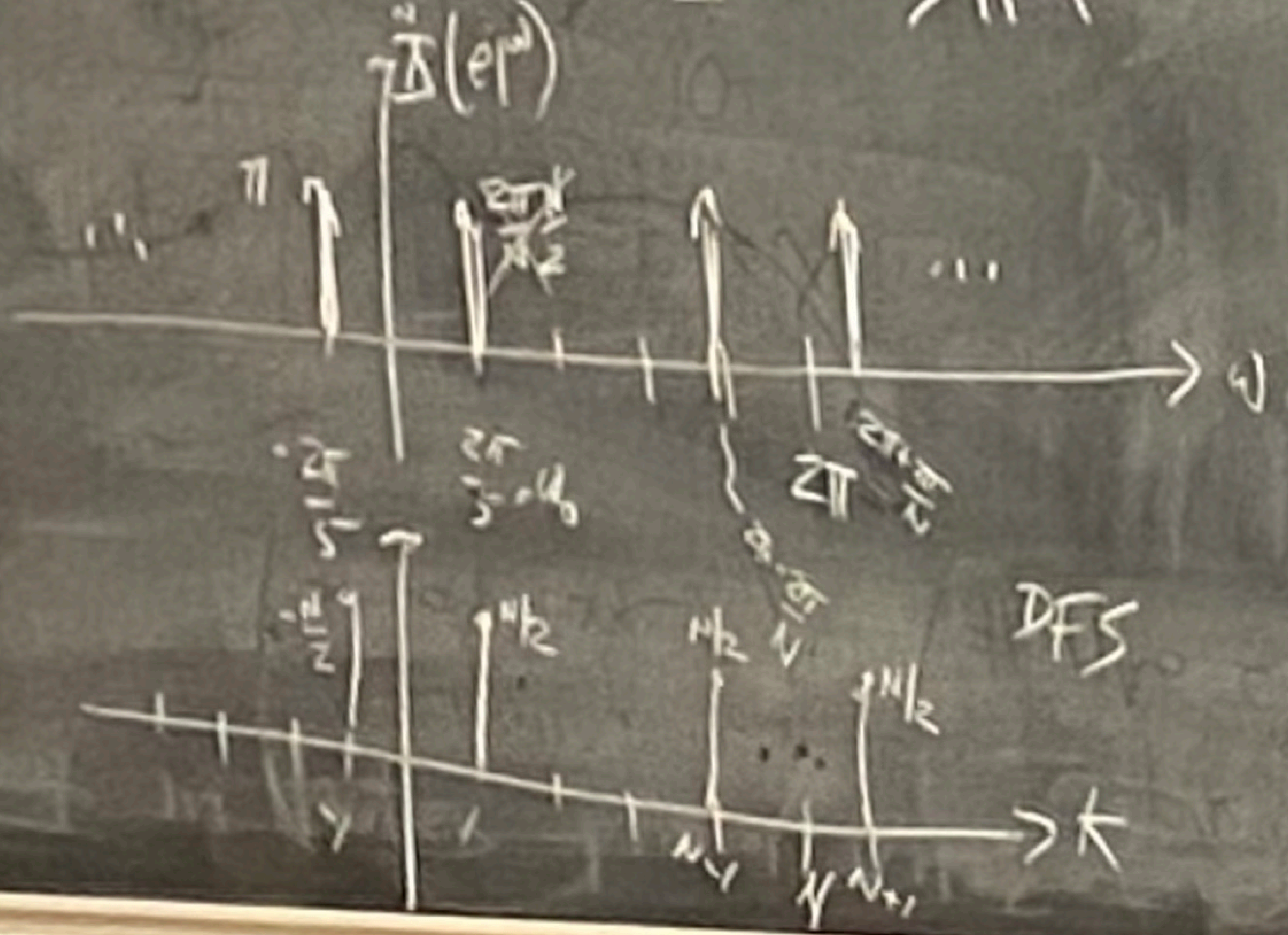
$N = 5$



Ex 1. $\tilde{x}[n] = \cos(\omega_0 n)$ $\omega_0 = \frac{2\pi}{N}$ N INTEGER

N=5 $= \frac{e^{j\omega_0 n}}{2} + \frac{e^{-j\omega_0 n}}{2}$

DTFT



DFS

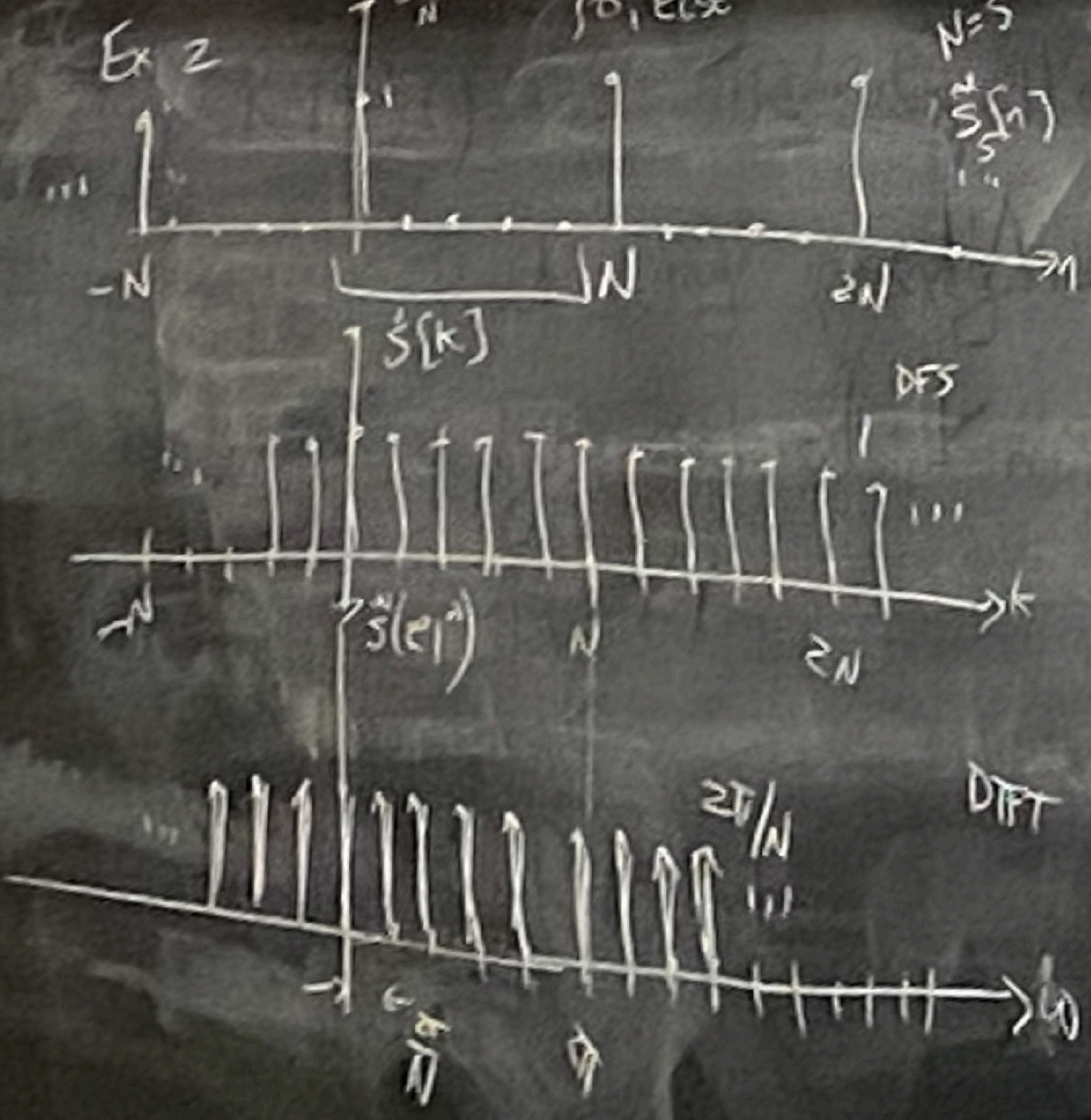
$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}[k]\, e^{jk\omega_0 n}$$

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n]\, e^{-jk\omega_0 n}$$

$$\frac{\tilde{X}[1]}{N} = \frac{1}{2}\ ; \quad \tilde{X}[1] = \frac{N}{2} = \cancel{\frac{2\pi}{2\omega_1}}\ ; \quad \tilde{X}[-1] = \frac{N}{2} = \tilde{X}[N-1]$$

Ex 2

$$\tilde{\delta}_N[n] = \begin{cases} 1, & n = rN \\ 0, & \text{else} \end{cases}$$



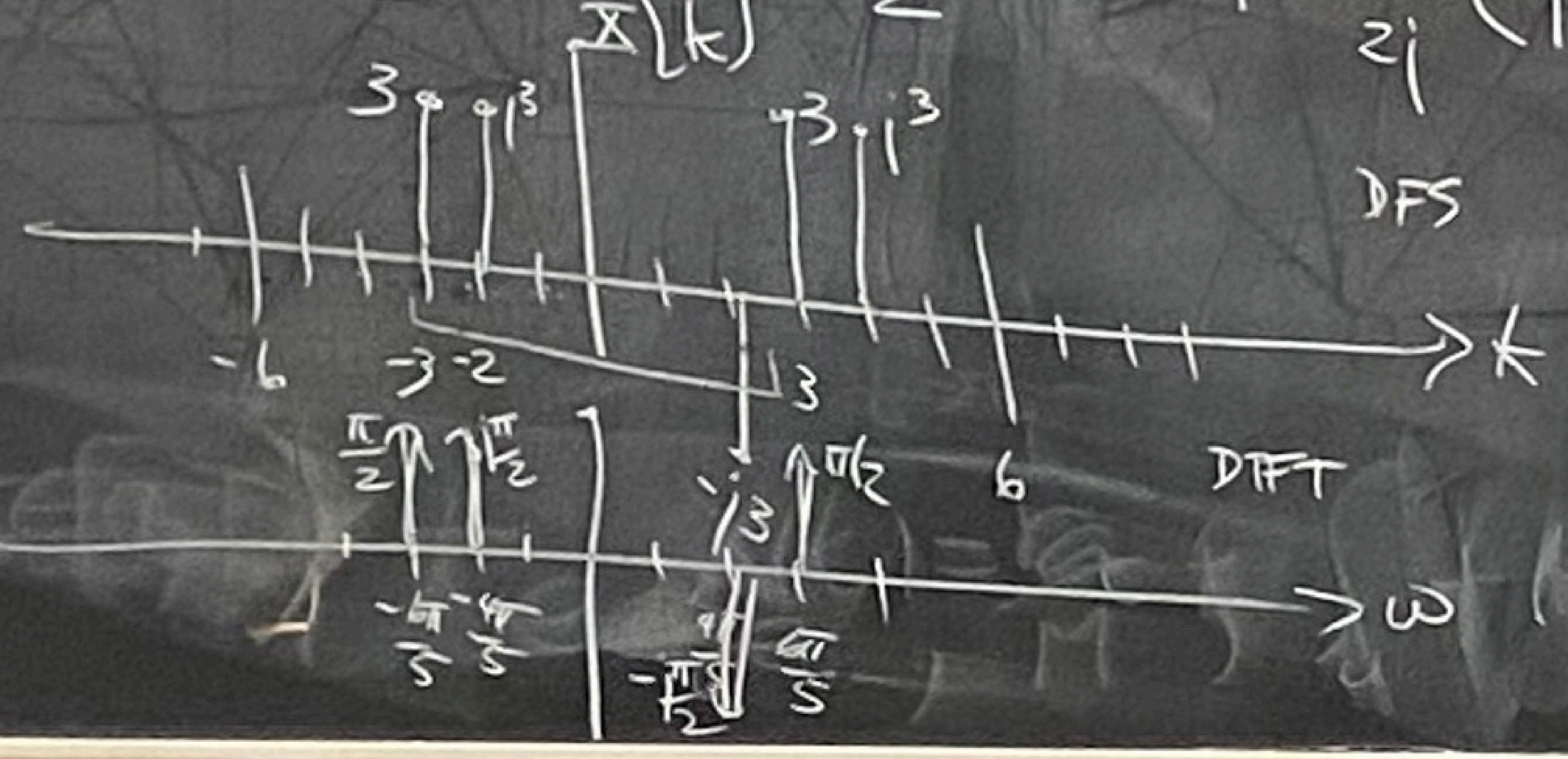
$0 \le k \le N-1$

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] e^{-jk\omega_0 n} = 1$$

Ex. 3: $\tilde{x}[n] = \cos\left(\frac{2\pi}{2} n\right) + \sin\left(\frac{2\pi}{3} n\right)$

$= \cos\left(\frac{(2\pi)(3)}{6} n\right) + \sin\left(\frac{2\pi}{6}\cdot(2)\, n\right)$

$= \frac{e^{j3\frac{2\pi}{6}n}}{2} + \frac{e^{-j3\cdot\frac{2\pi}{6}n}}{2} + \frac{1}{2j} e^{j2\frac{2\pi}{6}n} + \frac{-1}{2j} e^{-j2\cdot\frac{2\pi}{6}n}$



THE DFS AND THE DFT

(OS4P 8.3-8.6)

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DTFT

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

DTFS | DFS

$\tilde{x}[n] = \tilde{x}[n-N]$

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Also: $\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}[k]\, e^{jk\omega_0 n}$

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n]\, e^{-jk\omega_0 n}, \quad \omega_0 = \frac{2\pi}{N}$$

THE DISCRETE FOURIER TRANSFORM

FOR FINITE DURATION $x[n]$

ASSUME $x[n] = 0, \; n<0, \; n \geq N$

DFT

FINITE DURATION $x[n] = \begin{cases} \tilde{x}[n], & 0 \leq n \leq N-1 \\ 0, & \text{ELSE} \end{cases}$

$$\tilde{x}[n] = x[n \text{ MODULO } N] \triangleq x[((n))_N]$$

N-POINT DFT

IF $x[n] \neq \begin{cases} 0, & 0 \leq n \leq N-1 \text{ ONLY} \\ 0, & n<0, \; n \geq N \end{cases}$

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] \, e^{j\frac{2\pi n k}{N}}$$

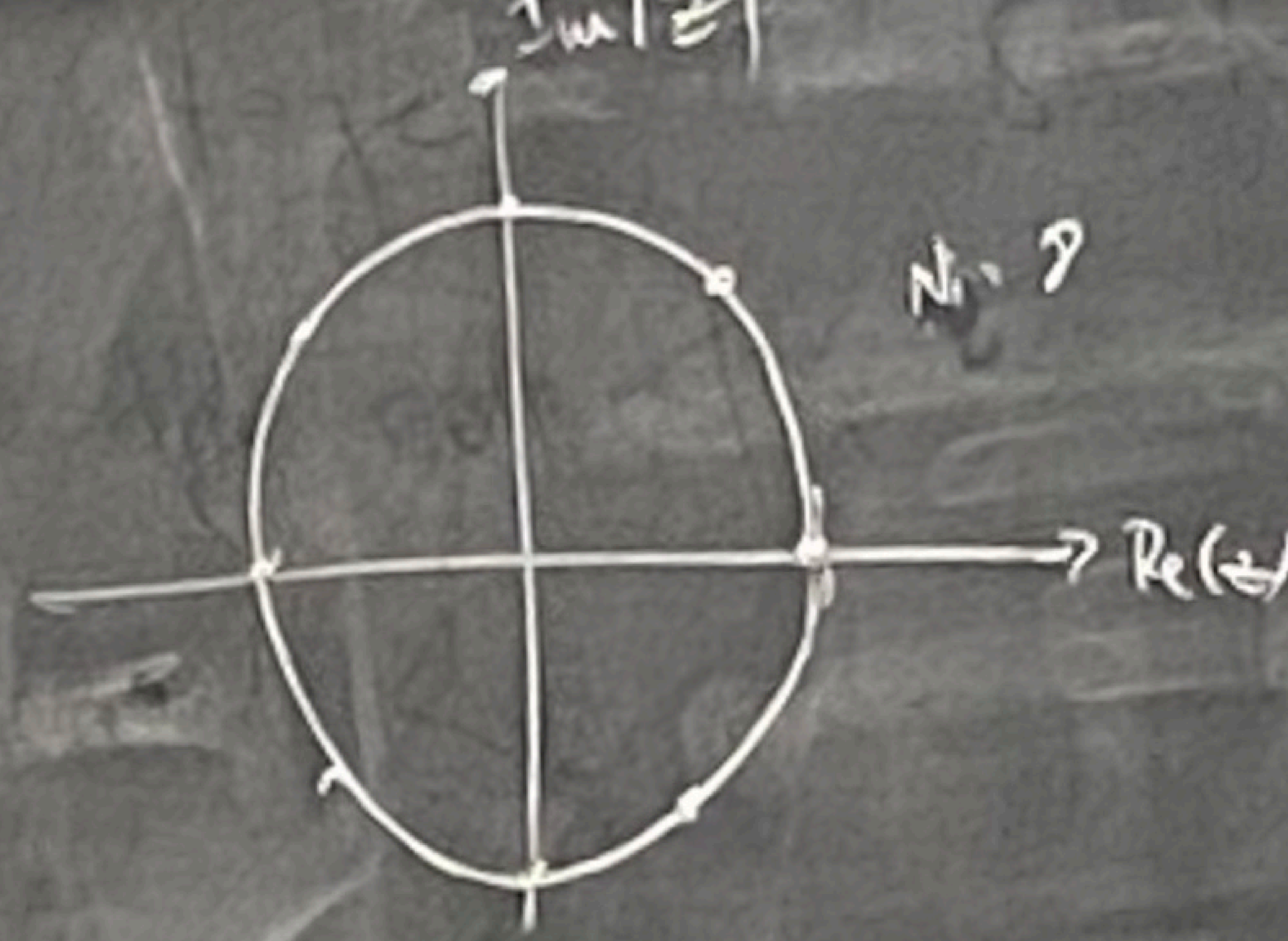
$$X[k] = \sum_{n=0}^{N-1} x[n] \, e^{-j\frac{2\pi n k}{N}}$$

COMPARING ZT, DTFT, DFT

ZT $X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$

DTFT $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$; $X(z)\big|_{z=e^{j\omega}}$

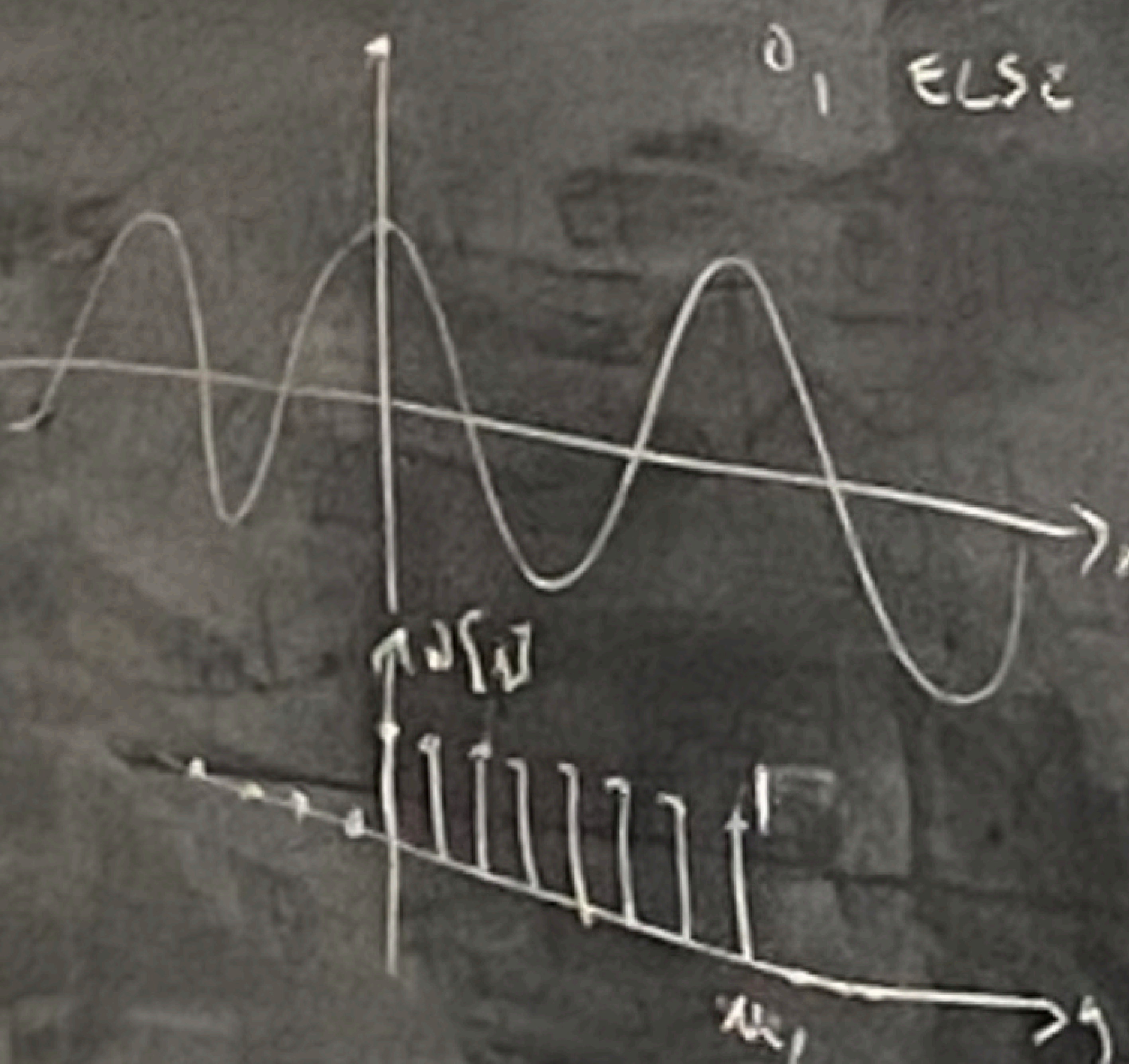
DFT $X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi k n}{N}}$; $X(z)\big|_{z=e^{j\frac{2\pi k}{N}}}$



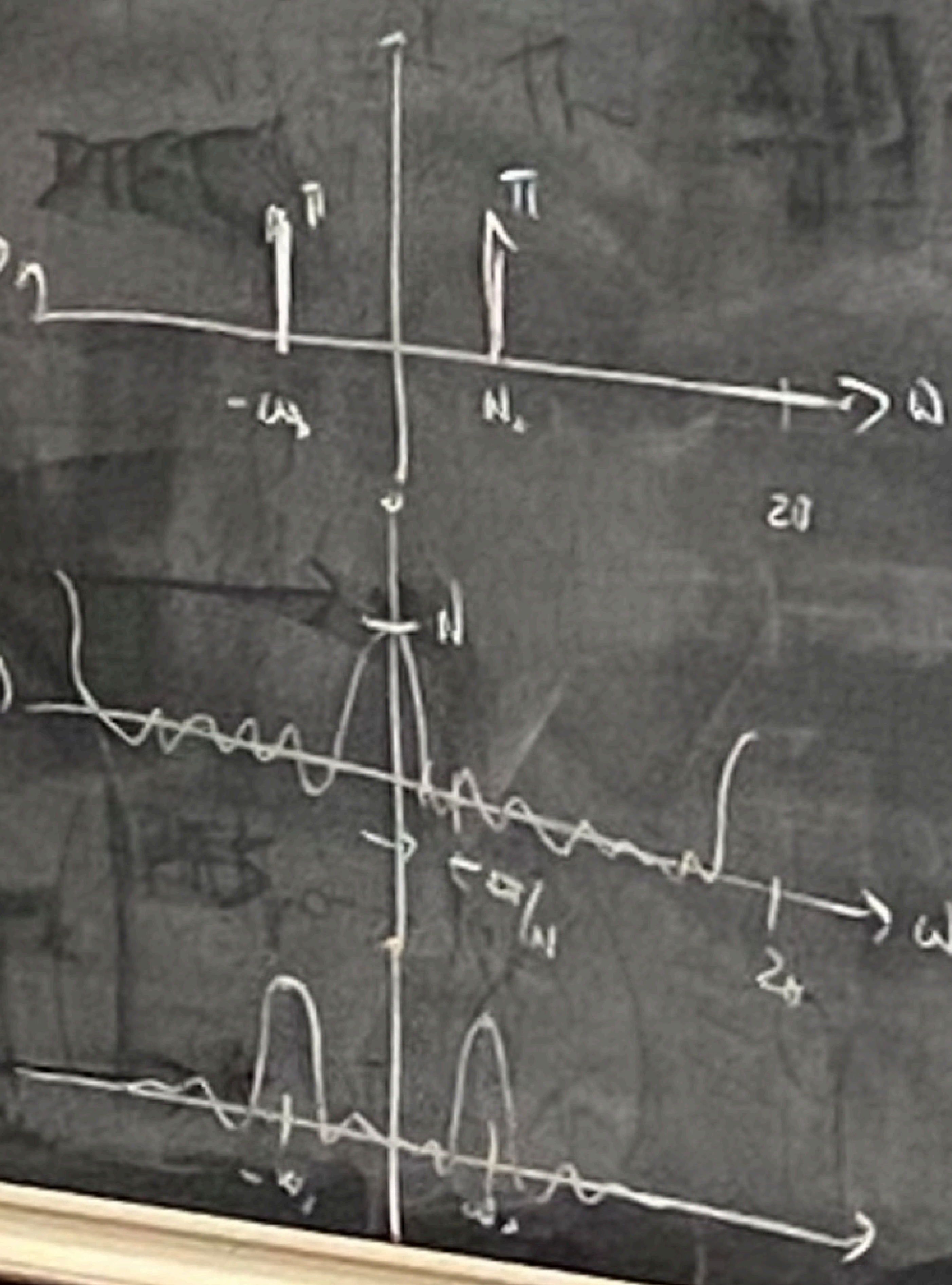
$\omega_0 \neq \frac{2\pi}{N}$

Ex. $x[n] = \cos(\omega_0 n), \quad 0 \le n \le N-1$

$0, \quad$ ELSE



DTFT



$$\frac{\sin\left(\frac{\omega N}{2}\right)}{\sin\left(\frac{\omega}{2}\right)} e^{-j\omega\frac{(N-1)}{2}}$$

(SELECTED)

PROPERTIES OF THE DFT

$$x[n] \overset{N}{\Longleftrightarrow} X[k]$$

$$W_N \triangleq e^{-j\frac{2\pi}{N}}$$

DFT $$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, W_N^{-nk}$$

$$X[k] = \sum_{n=0}^{N-1} x[n]\, W_N^{nk}$$

LINEARITY $$a x_1[n] + b x_2[n] \overset{N}{\Longleftrightarrow} a X_1[k] + b X_2[k]$$

TIME SHIFT $$x[((n-n_0))_N] \overset{N}{\Longleftrightarrow} e^{-j\frac{2\pi k n_0}{N}} X[k] = W_N^{k n_0} X[k]$$

MULT BY COMPLEX EXP.

$$x[n] W_N^{-rn} = x[n] e^{j\frac{2\pi}{N} r n} \Longleftrightarrow X[((k-r))_N]$$