2/21/24 INTRODUCTION TO THE DISCRETE FOURIER TRANSFORM (DFT) (O&SP 8.4–8.7)

DTFS

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]\,e^{j\frac{2\pi nk}{N}} = \frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]\,e^{jk\omega_0 n}$$

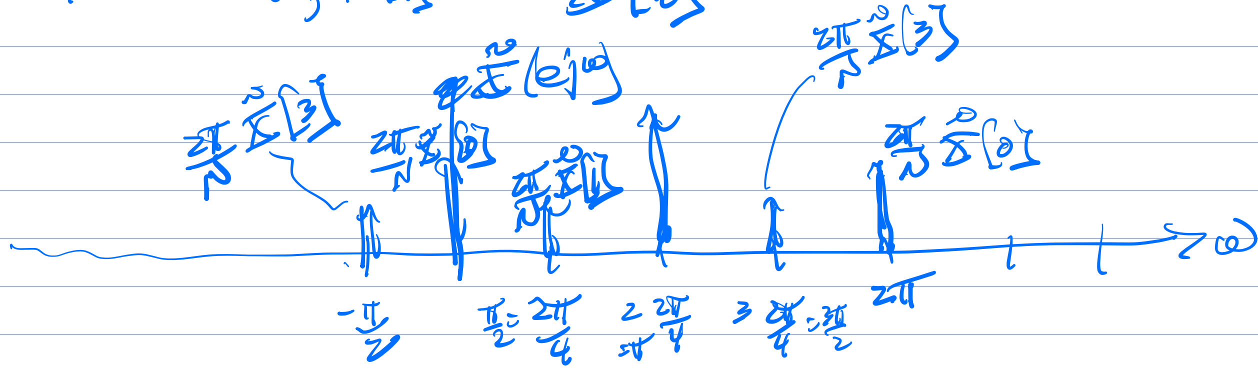
$$\omega_0 = \frac{2\pi}{N}$$

$$\tilde{X}[k] = \sum_{n=0}^{N-1}\tilde{x}[n]\,e^{-j\frac{2\pi nk}{N}} = \sum_{n=0}^{N-1}\tilde{x}[n]\,e^{-jk\omega_0 n}$$

$$\tilde{x}[n] \overset{\text{DTFT}}{\Longleftrightarrow} \tilde{X}(e^{j\omega}) = \sum_{r=-\infty}^{\infty}\sum_{\ell=0}^{N-1}\frac{2\pi}{N}\tilde{X}[\ell]\,\delta\!\left(\omega-\frac{2\pi\ell}{N}-2\pi r\right)$$

PERIOD N DTFS $\tilde{X}[k]$

Ex $N=4$, $\tilde{x}[n] \overset{\text{DTFS}}{\Longleftrightarrow} \tilde{X}[k]$



CONSIDER FINITE-DURATION FUNCTION $\tilde{x}[n]$ PERIOD N

$$x[n] \triangleq \begin{cases}\tilde{x}[n], & 0\le n\le N-1\\ 0, & \text{ELSE}\end{cases}$$

$$x[n] = \tilde{x}[n]\cdot w_R[n] \qquad w_R[n] = \begin{cases}1, & 0\le n\le N-1\\ 0, & \text{ELSE}\end{cases}$$

$$\tilde{x}[n] = x[((n))_N] \qquad ((n))_N \triangleq n \text{ MODULO } N$$

GIVEN FINITE-DURATION FUNCTION $x[n]$, $0 \le n \le N-1$

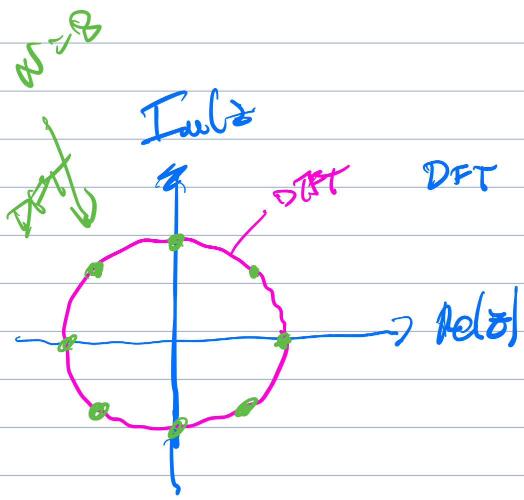
DFT (DISCRETE FOURIER TRANSFORM)

$$\boxed{\begin{aligned} x[n] &= \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, e^{j\frac{2\pi nk}{N}} \\ X[k] &= \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi nk}{N}} \end{aligned}}$$

COMPARING DFT, DTFT, ZT FOR $x[n] \neq 0$, $0 \le n \le N-1$

ZT $\quad X(z) = \sum_{n=0}^{N-1} x[n]\, z^{-n}$

DTFT $\quad X(e^{j\omega}) = \sum_{n=0}^{N-1} x[n]\, e^{-j\omega n} = X(z)\Big|_{z=e^{j\omega}}$



DFT $\quad X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi k n}{N}} = X(z)\Big|_{z=e^{j\frac{2\pi k}{N}}}$

$= X(e^{j\omega})\Big|_{\omega=\frac{2\pi k}{N}}$

SELECTED DFT PROPERTIES

CONSIDER $\quad x[n] \overset{N}{\Longleftrightarrow} X[k]$

LINEARITY $\quad a x_1[n] + b x_2[n] \overset{N}{\Longleftrightarrow} a X_1[k] + b X_2[k]$

CIRC. SHIFT IN TIME $\quad x[((n-n_0))_N] \overset{N}{\Longleftrightarrow} e^{-j\frac{2\pi n_0 k}{N}} X[k]$

MULT BY COMPLEX EXPONENTIAL $\quad x[n]\cdot e^{j\frac{2\pi r n}{N}} \Longleftrightarrow X[((k-r))_N]$

CONVOLUTION / MULTIPLICATION

CONSIDER $x_1[n] \overset{N}{\Longleftrightarrow} X_1[k]$

$$x_2[n] \overset{N}{\Longleftrightarrow} X_2[k]$$

Let $X[k] = X_1[k] \cdot X_2[k]$

$$y[n] \overset{N}{\Longleftrightarrow} X[k]$$

WHAT IS $y[n]$ IN TERMS OF $x_1[n], x_2[n]$?

Let $X_1[k] = \sum_{r=0}^{N-1} x_1[((r))_N]\, e^{-i\frac{2\pi r k}{N}}$

$$X_2[k] = \sum_{s=0}^{N-1} x_2[((s))_N]\, e^{-i\frac{2\pi s k}{N}}$$

$$y[n] = \frac{1}{N}\sum_{k=0}^{N-1} X_1[k] X_2[k]\, e^{i\frac{2\pi n k}{N}}$$

$$y[n] = \frac{1}{N}\sum_{k=0}^{N-1} \underbrace{\sum_{r=0}^{N-1} x_1[((r))_N]\, e^{-i\frac{2\pi r k}{N}}}_{X_1[k]} \underbrace{\sum_{s=0}^{N-1} x_2[((s))_N]\, e^{-i\frac{2\pi s k}{N}}}_{X_2[k]} e^{i\frac{2\pi n k}{N}}$$

$$y[n] = \sum_{r=0}^{N-1} x_1[((r))_N] \sum_{s=0}^{N-1} x_2[((s))_N]\, \frac{1}{N}\sum_{k=0}^{N-1} e^{i\frac{2\pi}{N}\underbrace{(n-s-r)}_{q}k}$$

Let $q = n-s-r$

$$\frac{1}{N}\sum_{k=0}^{N-1} e^{i\frac{2\pi}{N}qk} = 1, \; q=0$$

$$q \neq 0 \Rightarrow \frac{1}{N}\frac{1-e^{i\frac{2\pi}{N}qN}}{1-e^{i\frac{2\pi}{N}q}} = 0, \; q \neq 0$$

$= 1, \; s = n-r$ OR $q=0$

0, ELSE

$$\boxed{y[n] = \sum_{r=0}^{N-1} x_1[((r))_N]\, x_2[((n-r))_N]}$$

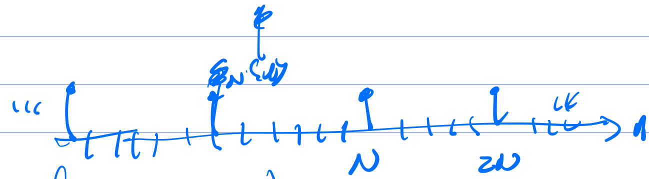
CIRCULAR CONVOLUTION

$x_1[((n))_N] \overset{N}{\circledast} x_2[((n))_N] \overset{N}{\Longleftrightarrow} X_1[k]\, X_2[k]$

WHEN IS $x_1[n] * x_2[n] = x_1[((n))_N] \overset{N}{\circledast} x_2[((n))_N]$

$$y[n] = \sum_{k=0}^{N-1} x_1[k]\, x_2[((n-k))_N] = x_1[n] * x_2[((n))_N]$$

$$x_2[((n))_N] = x_2[n] * s_N[n]$$



$$y[n] = x_1[n] \overset{N}{\circledast} x_2[((n))_N] = (x_1[n] * x_2[n]) * s_N[n]$$

Let $b[n] = x_1[n] * x_2[n]$

$$y[n] = \sum_{r=-\infty}^{\infty} b[n - rN]$$

$x_1[n]$ FINITE DURATION, LENGTH N_1

$x_2[n]$ FINITE DURATION, LENGTH N_2

$x_1[n] * x_2[n]$ FINITE DURATION LENGTH $N_1 + N_2 - 1$

LET $x_1[n] \overset{N}{\Longleftrightarrow} X_1[k]$

$x_2[n] \overset{N}{\Longleftrightarrow} X_2[k]$

$$X[k] = X_1[k] \cdot X_2[k]$$

$$y[n] \overset{N}{\Longleftrightarrow} X[k]$$

$$\boxed{y[n] = x_1[((n))_N] \overset{N}{\circledast} x_2[((n))_N] = x_1[n] * x_2[n] \quad \text{PROVIDED THAT } N \geq N_1 + N_2 - 1}$$