2/19/24 THE DTFS & THE DFT

DTFT of PERIODIC TIME FUNCTIONS

[OS4P 8.0-8.6]

$\boxed{\text{CTFT}}$ FOR $\int_{-\infty}^{\infty} |x(t)|^2 dt < \infty$ $\quad x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Omega)\, e^{j\Omega t}\, d\Omega$

$$X(j\Omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\Omega t}\, dt$$

$\boxed{\text{CTFS}}$ FOR $x(t) = x(t-T)$, $\forall t$

(PERIODIC W/ PERIOD T)

$$x(t) = \sum_{k=-\infty}^{\infty} X_k\, e^{j\frac{2\pi k t}{T}} = \sum_{k=-\infty}^{\infty} X_k\, e^{jk\Omega_0 t}$$

$\Omega_0 = \frac{2\pi}{T}$

$$X_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\, e^{-jk\Omega_0 t}\, dt$$

$\boxed{\text{DTFT}}$ IF $\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

IF $\tilde{x}[n]$ IS PERIODIC WITH PERIOD N

$$\tilde{x}[n] = \tilde{x}[n-N], \ \forall n, \text{ SOME } N$$

$\boxed{\text{DTFS}}$

LET $\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}[k]\, e^{j\frac{2\pi k n}{N}}$

k^{th} TERM $\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}[k]\, e^{jk\omega_0 n}$, $\omega_0 = \frac{2\pi}{N}$

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n]\, e^{-j\frac{2\pi n k}{N}} = \sum_{n=0}^{N-1} \tilde{x}[n]\, e^{-jk\omega_0 n}$$

CONSIDER $e^{j\frac{2\pi n k}{N}}$

$e^{j\frac{2\pi n(k+N)}{N}} = e^{j\frac{2\pi n k}{N}} \cancel{e^{j\frac{2\pi n N}{N}}}$ K+N TERM

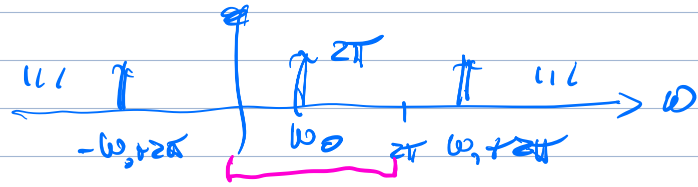
$\tilde{X}[k]$ PERIODIC W PERIOD N

THE DTFT OF PERIODIC TIME FUNCTIONS

DTFT :

$$x[n] = 1 \Longleftrightarrow X(e^{j\omega}) = 2\pi \sum_{r=-\infty}^{\infty} \delta(\omega - 2\pi r)$$

$$1 \cdot e^{j\omega_0 n} \Longleftrightarrow 2\pi = \sum_{r=-\infty}^{\infty} 2\pi\, \delta(\omega - \omega_0 - 2\pi r)$$



$$x[n] = \frac{1}{2\pi} \int_{0-\epsilon}^{2\pi-\epsilon} X(e^{j\omega}) e^{j\omega n}\, d\omega$$

$$= \frac{1}{\cancel{2\pi}} \int_{0-\epsilon}^{2\pi-\epsilon} \cancel{2\pi}\, \delta(\omega - \omega_0) e^{j\omega n}\, d\omega$$

$$= e^{j\omega_0 n}$$

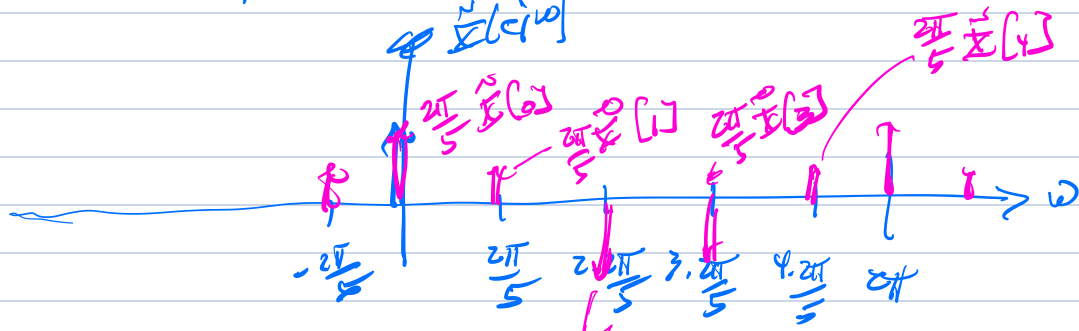
FOR PERIODIC TIME FUNCTIONS

$$\tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] e^{jk\omega_0 n} \overset{DTFT}{\Longleftrightarrow} \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] \sum_{r=-\infty}^{\infty} 2\pi\, \delta(\omega - k\omega_0 - 2\pi r)$$

$$\tilde{x}[n] \Longleftrightarrow \sum_{r=-\infty}^{\infty} \sum_{k=0}^{N-1} \frac{2\pi}{N} \tilde{X}[k]\, \delta(\omega - k\omega_0 - 2\pi r)$$

$\omega_0 = \frac{2\pi}{N}$

EX. $N = 5$
KNOW $\tilde{X}[0], \tilde{X}[1] \ldots \tilde{X}[4]$



For $\tilde{x}[n]$ period N

$\frac{2\pi}{5}\tilde{X}[2]$

$$\tilde{X}(e^{j\omega}) = \sum_{r=-\infty}^{\infty}\sum_{k=0}^{N-1}\frac{2\pi}{N}\tilde{X}[k]\,\delta(\omega - k\omega_0 - 2\pi r)$$

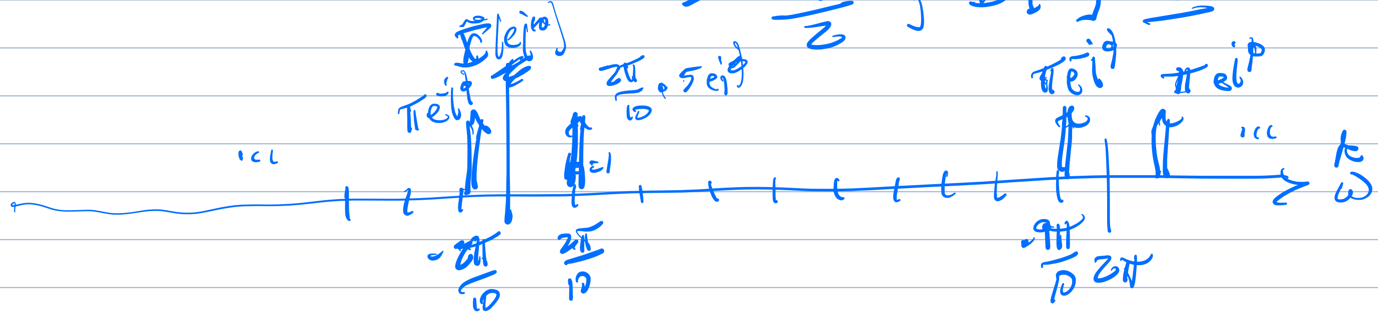
$\omega_0 = \frac{2\pi}{N}$

Examples

$$\tilde{x}(n) = \cos(\underbrace{.2\pi}_{\omega_0 = .2\pi} n + \phi) = \frac{e^{j\phi}}{2}e^{j.2\pi n} + \frac{e^{-j\phi}}{2}e^{-j.2\pi n}$$

$$\cos(.2\pi n + \phi) = \cos\left(\frac{2\pi n}{10} + \phi\right)$$

$N = 10$ $\quad \tilde{X}[1] = \frac{e^{j\phi}}{2}$, $\quad \tilde{X}[-1] = \frac{e^{-j\phi}}{2}$



$$\tilde{X}[k] = \sum_{n=0}^{9}\cos(.2\pi n + \phi) = \sum_{n=0}^{9} e^{j.2\pi n}\,\frac{e^{j\phi}}{2}\,e^{-j\frac{2\pi}{10}} + \underbrace{\quad}_{k=-1}$$

$k=1$

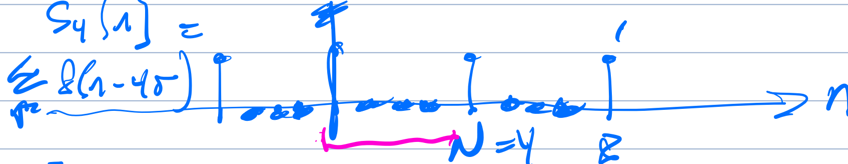
$\tilde{X}[1] = 5e^{j\phi}$

$\tilde{X}[-1] = 5e^{-j\phi}$

$$\frac{e^{j\phi}}{2}\sum_{n=0}^{9} e^{j\frac{2\pi}{10}n}\cdot e^{-j\frac{2\pi}{10}} = \frac{10e^{j\phi}}{2}$$

Ex. 2

$s_4[n] = \sum \delta(n-4r)$



$$\tilde{X}[k] = \sum_{n=0}^{3} s_4[n]\,e^{-jk\omega_0 n} = 1, \ \forall n$$

$\tilde{X}(e^{j\omega}) = \tilde{S}_4(e^{j\omega})$