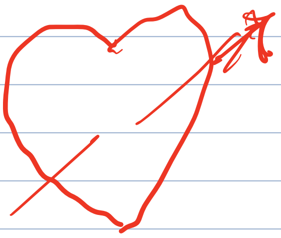
2/14/24 CHANGE OF SAMPLING RATE:

DECIMATION + INTERPOLATION

(AKA DOWN SAMPLING + UPSAMPLING)

[OYSP 4.6–4.7 PLUS ADSP NOTES]



REMINDER: QUIZ 1 2/28/24 (TWO WEEKS!)

CHANGE OF SAMPLING RATE:

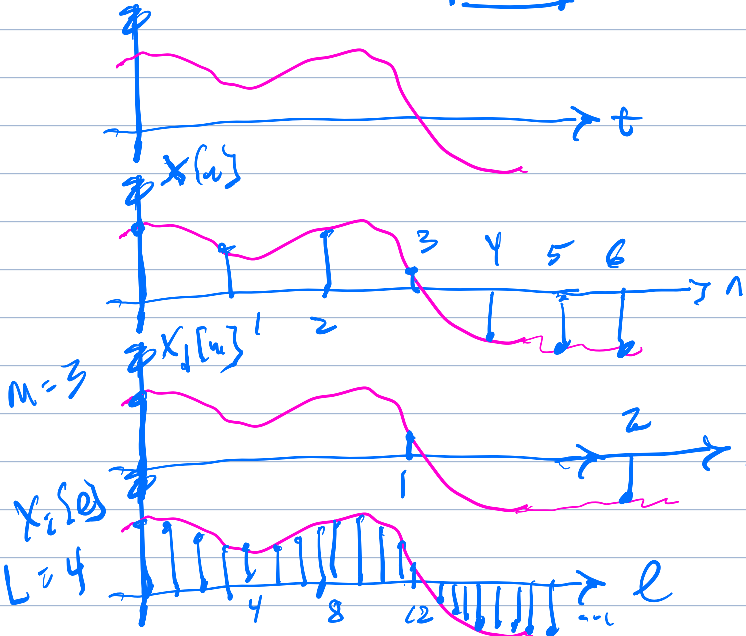
1. DECIMATION BY M — DOWN SAMPLING
2. INTERPOLATION BY L — UPSAMPLING
3. CHANGE IN RATE BY L/M

DECIMATION

$$x[n] \rightarrow \boxed{\downarrow M} \rightarrow x_d[m] = x[mM]$$

INTERPOLATION

$$x_c(t) \qquad x[n] \rightarrow \boxed{\uparrow L} \rightarrow x_i[\ell] = \begin{cases} x\left[\frac{\ell}{L}\right], & \frac{\ell}{L} \text{ INTEGER} \\ \text{SOMETHING ELSE}, & \text{OTHERWISE} \end{cases}$$



DECIMATION BY 3

INTERPOLATION BY 4

APPLICATIONS:

1. FRACTIONAL DELAYS
2. EFFICIENT FILTER DESIGN
3. COMPATIBILITY

MATHEMATICS OF DECIMATION

$$x[n] \longrightarrow \boxed{\downarrow M} \longrightarrow x_d[m] = x[mM]$$

$$\Updownarrow$$

$X(e^{j\omega})$ $\qquad\qquad$ $X_d(e^{j\omega'})$

EXPRESS $X_d(e^{j\omega'})$ IN TERMS of $X(e^{j\omega})$

WRONG METHOD:

$$X_d(e^{j\omega'}) = \sum_{m=-\infty}^{\infty} x_d[m] e^{-j\omega' m}$$

$$= \sum_{m=-\infty}^{\infty} x[mM] e^{-j\omega' m}$$

let $l = mM$
$m = \frac{l}{M}$

$$= \sum_{l=-\infty}^{\infty} x[l] e^{-j\omega' \frac{l}{M}} = \sum_{l=-\infty}^{\infty} x[l] e^{-j\frac{\omega'}{M} l}$$

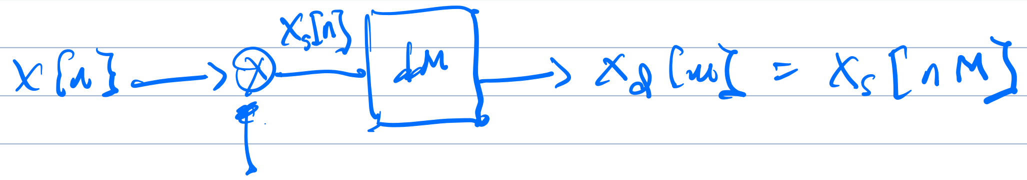
$$= X(e^{j\frac{\omega'}{L}})$$

CT: $x(t) \Longleftrightarrow X(j\Omega)$

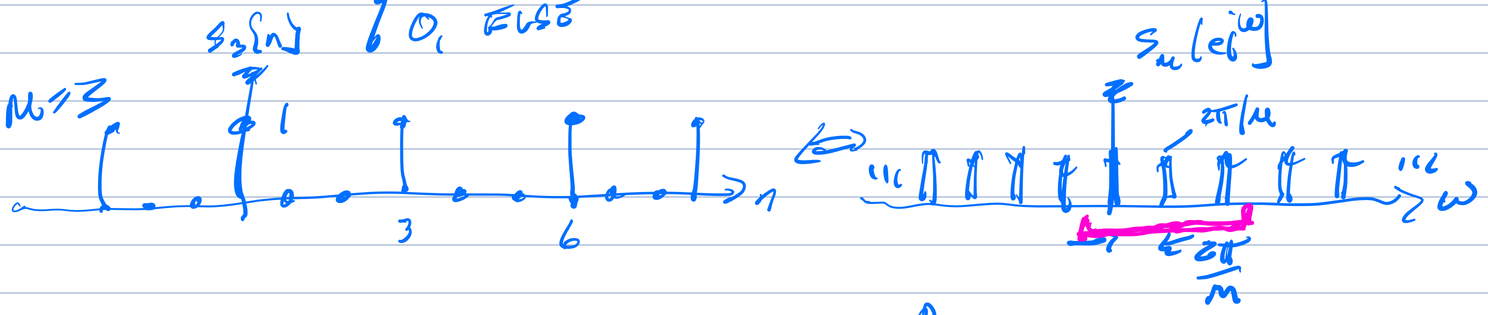
$x(at) \Longleftrightarrow \frac{1}{|a|} X\left(j\frac{\Omega}{a}\right)$

CHANGE RATE

CORRECT APPROACH:



$$s_M[n] = \begin{cases} 1, & n = rM \\ 0, & \text{ELSE} \end{cases}$$



$$S_M(e^{j\omega}) = \sum_{k=-\infty}^{\infty} \frac{2\pi}{M}\,\delta\!\left(\omega - \frac{2\pi k}{M}\right)$$

$$= \sum_{r=-\infty}^{\infty}\sum_{l=0}^{M-1} \frac{2\pi}{M}\,\delta\!\left(\omega - \frac{2\pi l}{M} - 2\pi r\right)$$

PROOF OF COTET

$$s[n] = \frac{1}{2\pi}\int_{0-\epsilon}^{2\pi-\epsilon} \underbrace{\frac{2\pi}{M}\sum_{l=0}^{M-1}\delta\!\left(\omega - \frac{2\pi l}{M}\right)}_{X(e^{j\omega})} e^{j\omega n}\,d\omega$$

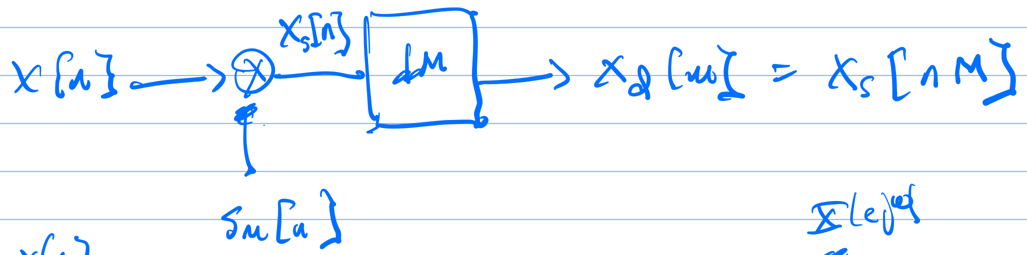
$$= \frac{1}{\cancel{2\pi}}\,\frac{\cancel{2\pi}}{M}\sum_{l=0}^{M-1}\underbrace{\int_{0-\epsilon}^{2\pi-\epsilon}\delta\!\left(\omega - \frac{2\pi l}{M}\right)e^{j\omega n}\,d\omega}$$

$$\frac{1}{M}\sum_{l=0}^{M-1} e^{j\frac{2\pi l n}{M}} = \frac{1}{M}\sum_{l=0}^{M-1}\left(e^{j\frac{2\pi n}{M}}\right)^{l}$$

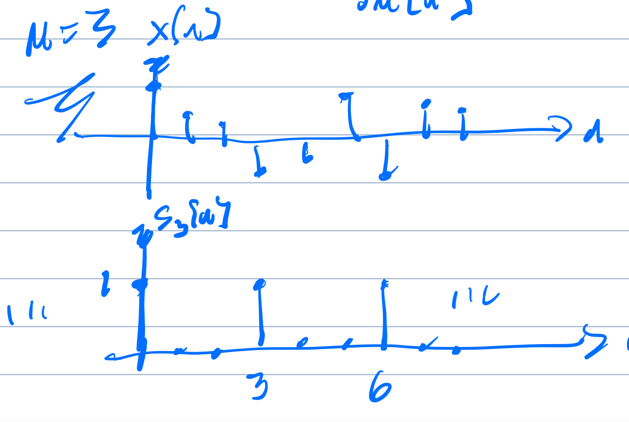
$$s[n] = \frac{1 - e^{j\frac{2\pi n \cancel{M}}{\cancel{M}}}}{1 - e^{+j2\pi n/M}}$$

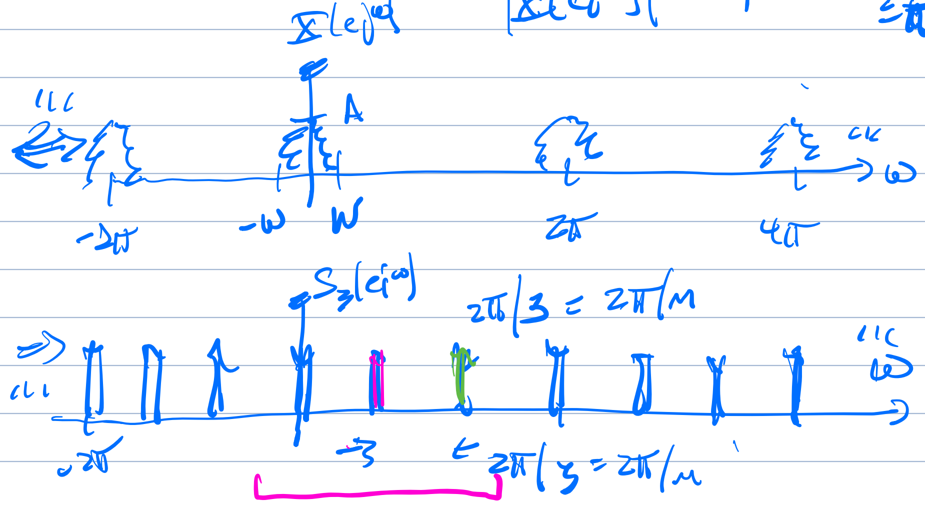
$$s[n] = \begin{cases} 1, & n = rM \\ 0, & n \neq rM \end{cases}$$

IMPACT OF DOWNSAMPLING ON FREQUENCY RESPONSE

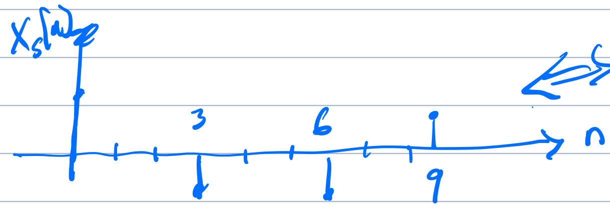


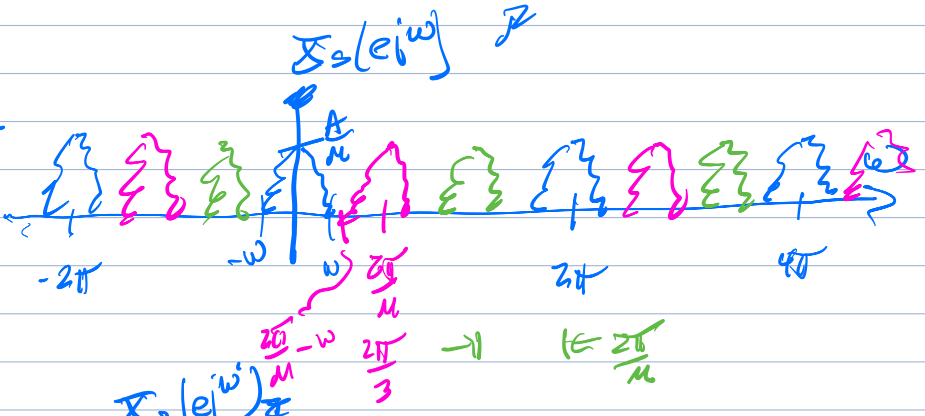
$|X(e^{j\omega})| = 0,\ W < |\omega| \leq \pi$

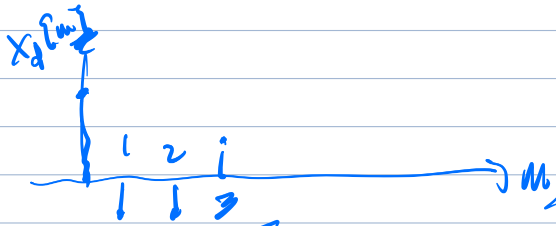


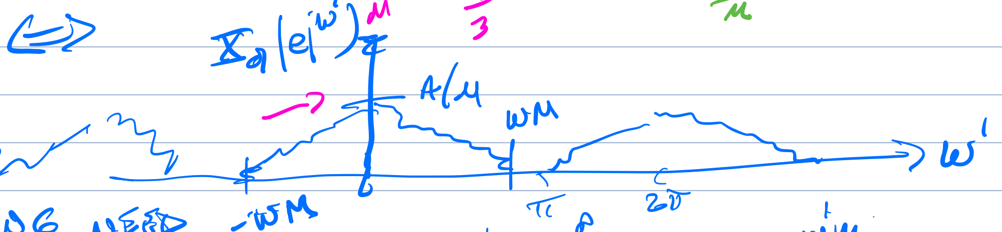


$x_s[n] = x[n] \cdot s_3[n] \Leftrightarrow X_s(e^{j\omega}) = \frac{1}{2\pi} X(e^{j\omega}) \circledast S_3(e^{j\omega})$









TO AVOID ALIASING NEED

$W < \frac{2\pi}{M} - W$

$2W < \frac{2\pi}{M}$

$$\boxed{W < \frac{\pi}{M}}$$

$$X_d(e^{j\omega'}) = \sum_{m=-\infty}^{\infty} x_d[m] e^{-j\omega' m}$$

$$= \sum_{m=-\infty}^{\infty} x_s[mM] e^{-j\omega' m}$$

let $l = mM$

$m = \frac{l}{M}$

$$= \sum_{l=-\infty}^{\infty} x_s[l] e^{-j\frac{\omega'}{M} l}$$

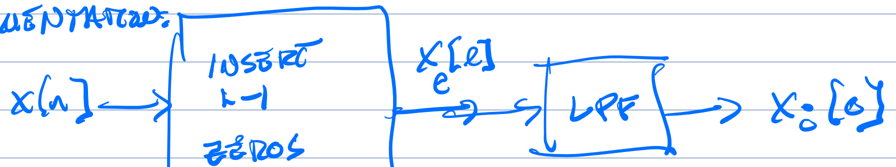
$$= X_s(e^{j\frac{\omega'}{M}})$$

NOTE: SPREADING BY M CAUSES AREAS OF DELTA FUNCTIONS TO BE MULTIPLIED BY M

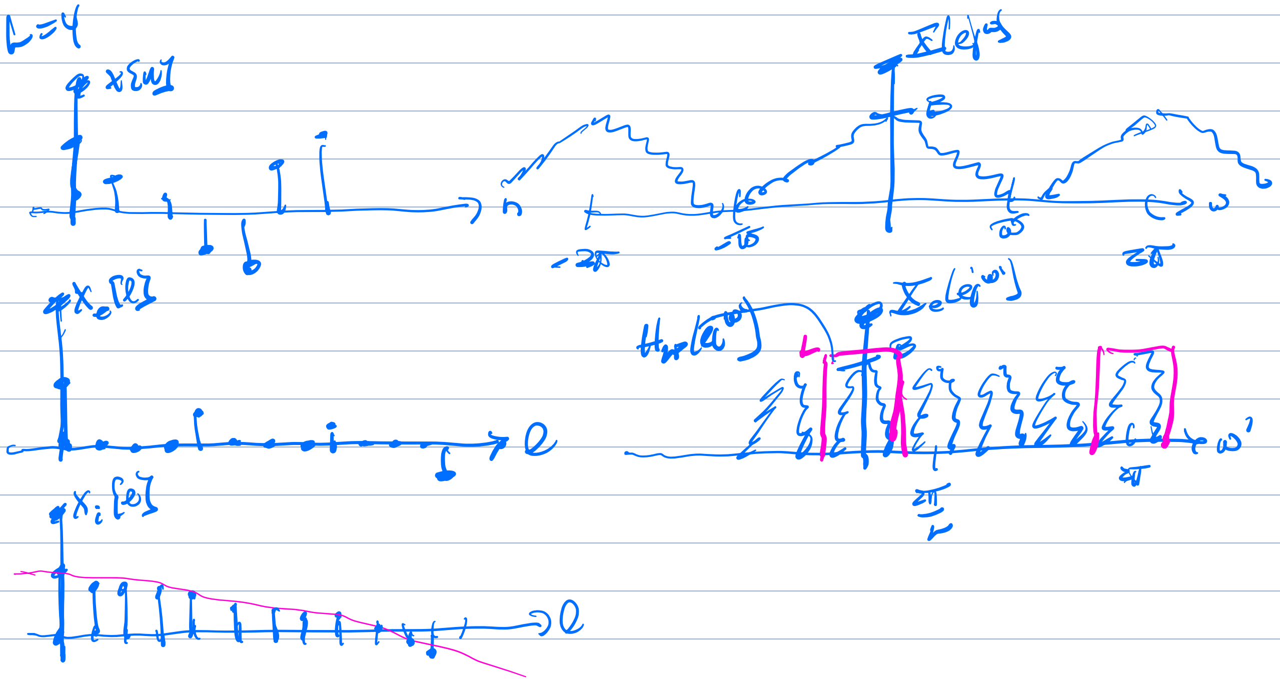
DISCRETE-TIME INTERPOLATION

$$x[n] \rightarrow \boxed{\uparrow L} \rightarrow x_i[l] = x\left[\frac{l}{L}\right], \quad \frac{l}{L} \text{ INTEGER}$$

IMPLEMENTATION:



$$X_e(e^{j\omega'}) = X(e^{j\omega' L})$$



$$H_{LP}(e^{j\omega'}) = \begin{cases} L, & |\omega'| < \frac{\pi}{L} \\ 0, & \frac{\pi}{L} < |\omega'| \le \pi \end{cases}$$

TO BE CONTINUED ON FRIDAY:

1. CONCLUSION OF UPSAMPLING
2. CHANGE OF SAMPLING RATE