18-290
Signals and Systems
Profs. Aswin Sankaranarayanan and Richard Stern — Spring 2017

Working with the Delta Function $\delta(t)$

Note: We briefly introduced the topic of delta functions in the lecture of January 24. These notes will review the discussion from that lecture.

The unit impulse function $\delta(t)$ has a long and honorable history in signal processing. In its classic form the unit impulse function is used to represent pulse-like signals that are very brief compared to any of the meaningful time constants of a realizable system. It is *much* easier performing these computations using the idealized delta functions than with the original brief signals, even though we must put up with some mathematical extremes. We will being by discussing three equivalent definitions of the delta function. Following that we will comment on how to obtain results for computations that involve the delta function.

Axiomatic definition of the delta function.

In many engineering courses, the delta function is defined in the following fashion:

$$\delta(t) = 0, \text{ for } t \neq 0 \tag{1}$$

$$\int_{-\infty}^{\infty} \delta(t)dt = 1 \tag{2}$$

Since the delta function equals zero by definition for values of t other than zero, it must have infinite amplitude at $t = 0$ in order for it to maintain an area of one at $t = 0$. So under these circumstances we may think of the delta function as being infinitesimally wide but infinitely tall, with unit area.

Limiting definitions of the delta function.

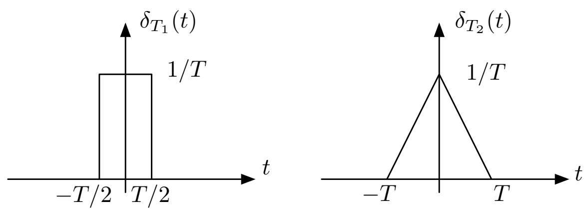

Figure 1: Two functions that approach $\delta(t)$ in the limiting case.

Consider a function $\delta_T(t)$ which has finite amplitude and width, but unit area. Two examples of such functions, $\delta_{T_1}(t)$ and $\delta_{T_2}(t)$, are depicted in the figure above. In either case, we can express the delta function as the limit

$$\delta(t) = \lim_{T \to 0} \delta_T(t) \tag{3}$$

where again, $\delta_T(t)$ can be any function of t that in the limit as T goes to zero has infinitesimal width and infinite height with unit area. In practice, the shape of the function does not matter, provided that the area remains constant independent of T as is the case with the functions $\delta_{T_1}(t)$ and $\delta_{T_2}(t)$ depicted above. It can easily be seen that this definition is consistent with (and in fact is a generalization of) the first definition.

Implicit definition of the delta function.

The most general definition of the delta function, which we encourage you to use always, is the so-called distributional definition of the delta function. Specifically, let the function $\phi(t)$ be any function of t that is continuous everywhere. The delta function is then defined as

$$\int_{-\infty}^{\infty} \delta(t-a)\phi(t)dt = \phi(a) \tag{4}$$

Please note that this is a different kind of definition for a function than you may be used to: $\delta(t)$ is not defined by what it *is* but rather by what it *does* when subjected to the very specific operations of multiplication by a continuous "testing" function and than integration over all time. This type of definition is sometimes referred to as an *implicit* rather than explicit definition. In our work with delta functions, we will *only* work with them in the context of multiplication by a continuous function followed by subsequent integration. In our work, the integration operation will be in the context of either convolution or Fourier transformation. (For your information, convolution enables us to determine the output of a linear time-invariant system given the system's input and its "impulse response," which in fact is the response of a system to the input $\delta(t)$. Fourier transformation is the basic operation that enables us to convert a time function directly to its equivalent representation in the frequency domain. Both operations will be extremely important to us in the weeks to come.)

It is easy to see that this definition is consistent with Eqs. (1) and (2). Specifically, $\delta(t)$ must equal 0 for (in this case) $\phi \neq a$ because the result of the integral depends only on the value of $\phi(t)$ at $t = a$. Since we have no idea what $\phi(t)$ is equal to (and in principle it could be nonzero everywhere), the fact that $\int_{-\infty}^{\infty} \delta(t-a)\phi(t)dt = \phi(a)$ implies that $\delta(t)$ equals zero everywhere except for $t = a$. Equation (4) also reduces to Eq. (2) if we let $a = 0$ and $\phi(t) = 1$ for all t.

The delta function is also sometimes referred to as a "sifting function" because it extracts the value of a continuous function at one point in time.

Computation with the delta function.

We encourage you to approach the evaluation of all integrals involving the delta function using the procedure implied by Eq. (4). Specifically, evaluate the integral by applying three-step procedure:

1. Ask the question "What variable is being integrated?" [t in Eq. (4)]
2. Ask the question "What is the value of that variable that causes the argument of the delta function to equal zero?" [$t = a$ in Eq. (4)]
3. Then the result of the integration is the rest of the integrand evaluated at that value of the variable that is being integrated. [$\phi(a)$ in Eq. (4)]

We will illustrate these principles in a few examples below.

Example 1

$$\int_{-\infty}^{\infty} x(\tau)\delta(t - \tau - a)d\tau \tag{5}$$

This is an equation that may come up in a convolution problem when the system is an ideal delayer. The evaluation of the integral is straightforward following the discussion above:

1. "What variable is being integrated?" [τ in Eq. (5)]
2. "What is the value of that variable that causes the argument of the delta function to equal zero?" [$\tau = t - a$ in Eq. (5)]
3. The result of the integration is the rest of the integrand evaluated at that value of the variable that is being integrated. [$x(t - a)$ in Eq. (5)]

Example 2

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega - \omega_0)e^{j\omega t}d\omega \tag{6}$$

This is an example of an inverse continuous-time Fourier transform, but the evaluation once again is straightforward:

1. "What variable is being integrated?" [ω in Eq. (6)]

2. "What is the value of that variable that causes the argument of the delta function to equal zero?" [$\omega = \omega_0$ in Eq. (6)]

3. The result of the integration is the rest of the integrand evaluated at that value of the variable that is being integrated. [$\frac{1}{2\pi}e^{j\omega_0 t}$ in Eq. (6)]

Example 3

$$\int_{-\infty}^{\infty} \delta(2t)dt \tag{7}$$

This integral, which illuminates a property of delta functions, is only slightly less straightforward. In principle, we cannot evaluate this integral directly because Eq. (4) is defined in terms of $\delta(t)$ rather than $\delta(2t)$. Nevertheless, we can easily work around this issue with a change of variables. Specifically, let $t' = 2t$. Then $dt' = 2dt$, while $t = t'/2$ and $dt = dt'/2$. Hence we can write directly

$$\int_{-\infty}^{\infty} \delta(2t)dt = \int_{-\infty}^{\infty} \delta(t')dt'/2 = \frac{1}{2}\int_{-\infty}^{\infty} \delta(t')dt' = \frac{1}{2}$$

This last result makes sense, as replacing the argument t in the delta function by $2t$ causes the delta function to be compressed by a factor of 2 in time. Consequently the area of the delta function will be multiplied by a factor of 1/2.

Again, we restate that *every* integral involving delta functions can (and should!) be evaluated using the three-step procedure outlined above.

The unit step function and derivatives of discontinuous functions

As you know, the continuous-time unit step function is defined as

$$u(t) = \begin{cases} 0 & t < 0, \\ 1 & t > 0 \end{cases} \tag{8}$$

(We do not need to worry about the definition of $u(0)$ for now or for that matter, ever). The unit step function can be considered to be the integral of the delta function in that

$$u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau \tag{9}$$

While this may imply that $\delta(t)$ is the derivative of $u(t)$, this cannot be stated in the ordinary sense because of the discontinuity of $u(t)$ at $t = 0$. Nevertheless, we can use delta functions to represent the derivatives of functions that are continuous except for a finite number of points. For example, if $x(t)$ is continuous everywhere except for $t = a$, and $x(a+) = x(a-) + k$, then the derivative of $x(t)$ would be

$$\frac{dx}{dt} = \begin{cases} \frac{dx}{dt} \text{ in the ordinary sense for } t \neq a, \\ k\delta(t-a) \text{ for } t = a \end{cases} \tag{10}$$

In other words, if there are isolated discontinuities in an $x(t)$ that is otherwise continuous, the derivative of $x(t)$ would be the ordinary derivative where $x(t)$ is continuous, and there would be delta functions at the locations along the t axis where the discontinuities are observed. The areas of these delta functions would be equal to the size of the discontinuity at that location.

For example, if

$$x(t) = \begin{cases} t^2 & t < 3, \\ t^2 + 2 & t > 3 \end{cases}$$

then we would observe

$$\frac{dx(t)}{dt} = 2t + 2\delta(t-3)$$

Similarly, if

$$u(t) = \begin{cases} 0 & t < 0, \\ 1 & t > 0 \end{cases}$$

then we would observe

$$\frac{du(t)}{dt} = \delta(t) \tag{11}$$