Digital Signal Processing (18-491/18-691)
Spring Semester, 2023

Allpass, Minimum Phase, and Linear Phase Systems

I. Introduction

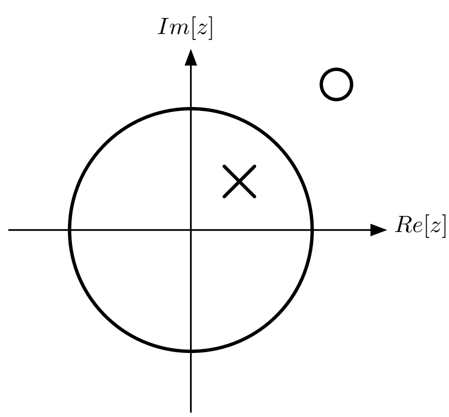
In previous lectures we have discussed how the pole and zero locations determine the magnitude and phase of the DTFT of an LSI system. In this brief note we discuss three special cases: allpass, minimum phase, and linear phase systems.

II. Allpass systems

Consider an LSI system with transfer function

$$H(z) = \frac{z^{-1} - a^*}{1 - az^{-1}}$$

Note that the system above has a pole at $z = a$ and a zero at $z = (1/a)^*$. This means that if the pole is located at $z = re^{j\theta}$, the zero would be located at $z = (1/r)e^{j\theta}$. These locations are illustrated in the figure below for $r = 0.6$ and $\theta = \pi/4$.



As usual, the DTFT is the z-transform evaluated along the unit circle:

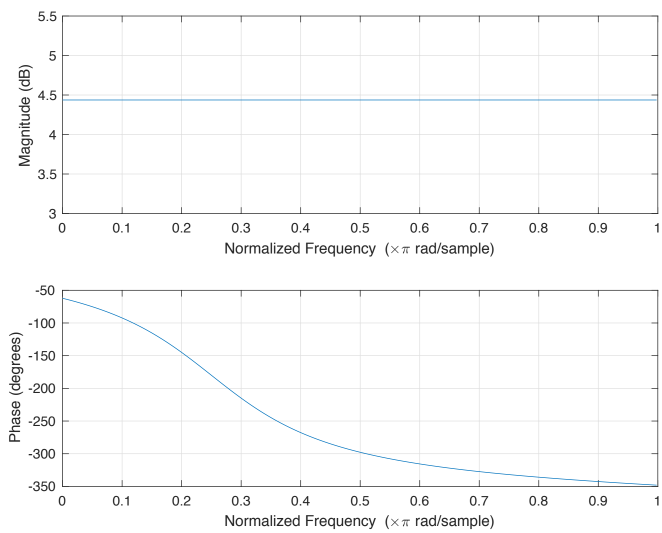
$$H(e^{j\omega}) = H(z)|_{z=0}$$

It is easy to obtain the squared magnitude of the frequency response by multiplying $H(e^{j\omega})$ by its complex conjugate:

$$\begin{aligned}\left|H(e^{j\omega})\right|^2 &= H(e^{j\omega})H^*(e^{j\omega}) = \frac{(e^{-j\omega} - re^{-j\omega})(e^{j\omega} - re^{j\omega})}{(1 - re^{j\theta}e^{-j\omega})(1 - re^{-j\theta}e^{j\omega})} \\ &= \frac{(1 - re^{j(\theta-\omega)} - re^{-j(\theta-\omega)} + r^2)}{(1 - re^{j(\theta-\omega)} - re^{-j(\theta-\omega)} + r^2)} = 1\end{aligned}$$

Because the squared magnitude (and hence the magnitude) of the transfer function is a constant independent of frequency, this system is referred to as an *allpass* system. A sufficient condition for the system to be allpass is for the poles and zeros to appear in "mirror-image" locations as they do in the pole-zero plot on the previous page. (More formally the pole and zero are located in *conjugate reciprocal* locations.)

The frequency response of the allpass system with complex impulse response with a pole at $z = 0.6e^{jpi/4}$ and a zero at $z = \left(\frac{1}{0.6}\right)e^{j\pi/4}$ is depicted below:



Additional comments:

- If $h[n]$ is real, the poles and zeros would each occur in complex-conjugate pairs. We note that even though the magnitude is constant, the phase does depend on frequency.
- This property implies that you can invert the magnitude of a pole or zero location in an LSI system without changing the magnitude of the transfer function of that system.

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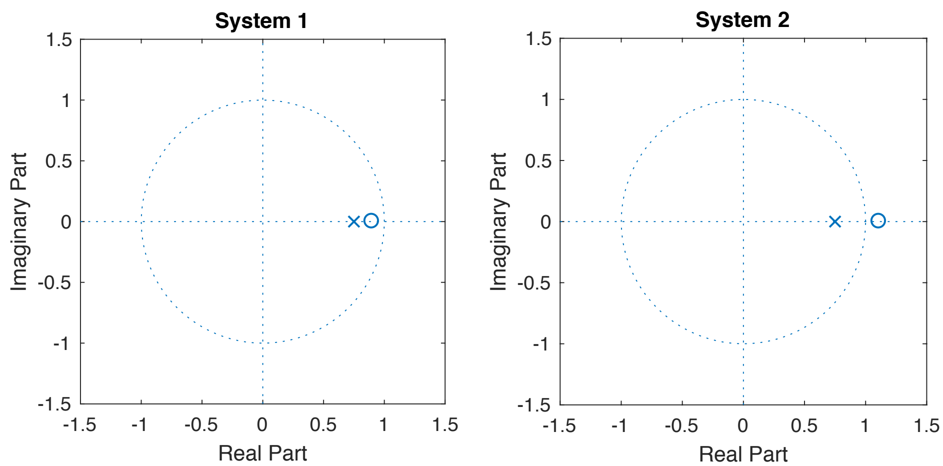
III. Minimum-phase systems

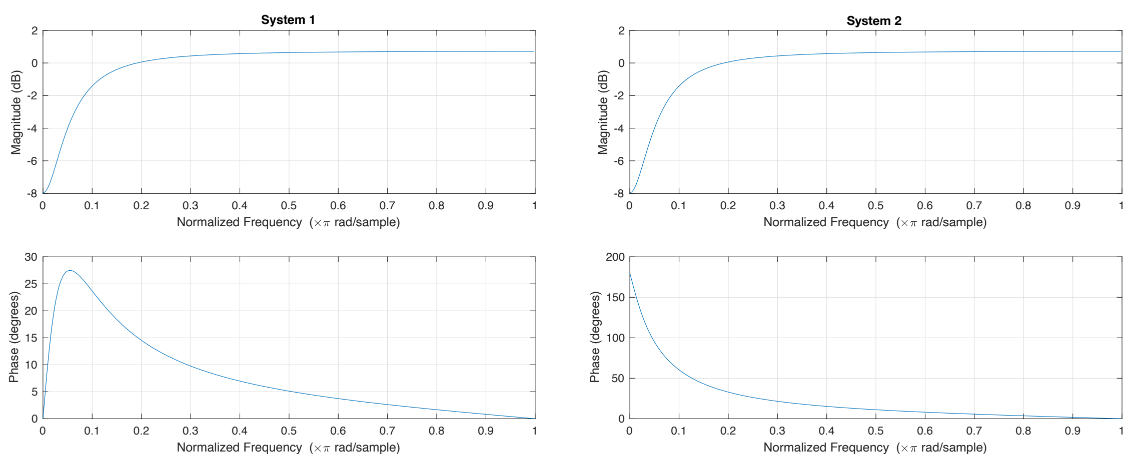
Now let us consider an example of reflecting a zero inside the unit circle. Specifically, consider the two LSI systems:

$$H_1(z) = \frac{z - \frac{9}{10}}{z - \frac{3}{4}} \text{ and } H_2(z) = \frac{z - \frac{10}{9}}{z - \frac{3}{4}}$$

The pole-zero plots for these two systems are depicted below:



We note that the two systems should have the same magnitude of the frequency response, as System 1 can be obtained by cascading System 2 with an allpass filter with a zero at $z = 9/10$ and a pole at $z = 10/9$. And (unsurprisingly) this is the case:



Note that the magnitude of the phase shift is much greater for System 2 than it is for System 1. In general System 1 is preferred over System 2, because it has less phase lag, which is normally considered to be desirable. In the general case, a system is considered to be *minimum phase* if all

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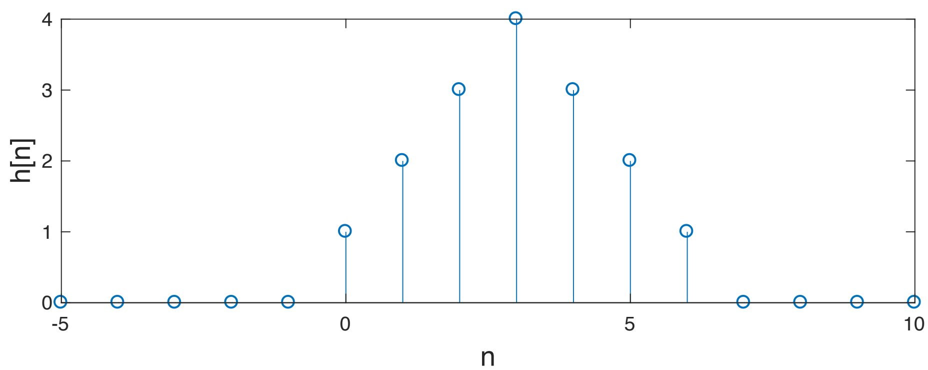
of its zeros as well as all poles are inside the unit circle. (Recall that you want to have all poles inside the unit circle so that a system can be both causal and stable.) We also note that a system is considered to be *maximum phase* if all poles and zeros are outside the unit circle, although this is not a very interesting case in practice.

IV. Linear-phase systems

Consider the causal system with the triangularly-shaped impulse response depicted below:



Note that this response is symmetric about the point $n = 3$. We can think of this $h[n]$ as a sample response in the form of $h[n] = h'[n-3]$ where the sample response $h'[n]$ is a real and even discrete-time function. This means that $H(e^{j\omega})$, the DTFT of $h[n]$, would be the product of $H'(e^{j\omega})$, which would be a real and even function of frequency and hence zero phase, and the term $e^{-j3\omega}$, which is the linear phase shift produced by the delay of $h[n]$ by three samples relative to $h'[n]$. Hence $H(e^{j\omega})$, the DTFT of $h[n]$, would be *linear phase*. In general a system is linear phase if its sample response is symmetric about its midpoint.

Now lets examine the impact of the linear-phase constraint on the pole-zero locations of a system. As an example, consider the transfer function of the system depicted above:

$$H(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 3z^{-4} + 2z^{-5} + 1z^{-6}$$

Let's multiply the numerator and denominator of $H(z)$ by z^6:

$$H(z) = \frac{z^6 + 2z^5 + 3z^4 + 4z^3 + 3z^2 + 2z + 1}{z^6}$$

The zeros of $H(z)$ are obtained (as usual) by setting the numerator of the expression above to zero:

$$z^6 + 2z^5 + 3z^4 + 4z^3 + 3z^2 + 2z + 1 = 0$$

Because of the symmetric form of $h[n]$, we also note multiplying the above equation by z^6 produces

$$1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 3z^{-4} + 2z^{-5} + 1z^{-6} = 0$$

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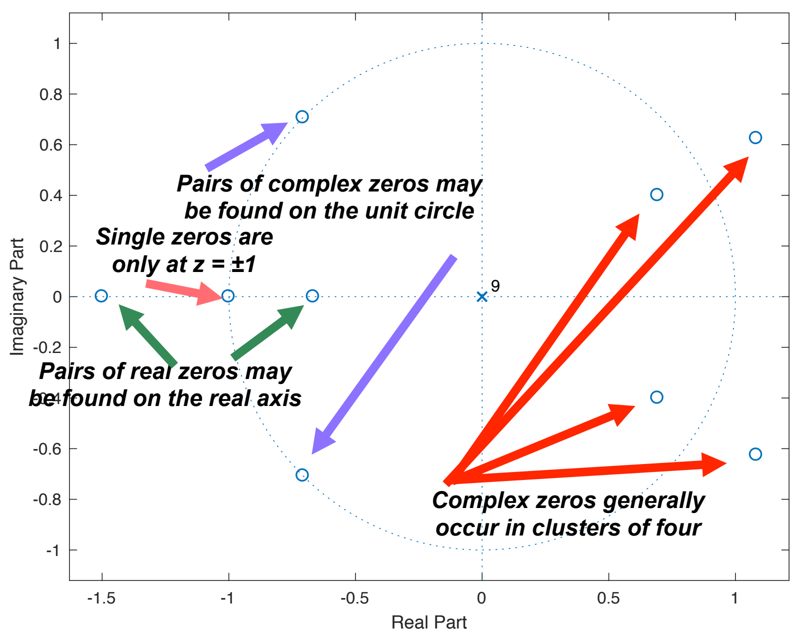
The above two equations tell us that if a particular value of z is a zero of the linear-phase system described by $H(z)$, z^{-1}, the inverse value of z would also be a zero of $H(z)$. Now if a particular zero is located at an arbitrary location $z = re^{j\theta}$, its inverse would be located at $z^{-1} = (1/r)e^{-j\theta}$. In other words, the reciprocal location would be found at the reciprocal of the original magnitude and the *negative* of the original phase shift. Of course, if $h[n]$ is real, the zeros would appear in complex-conjugate pairs, which would be accompanied by their reciprocal complex-conjugate locations. The diagram below summarizes the possible locations of zeros in a finite-impulse response linear-phase system:



Two final comments on linear-phase systems:

- Infinite-duration sample response can also have linear phase. They just cannot also be causal. For example, $h[n] = \left(\frac{1}{2}\right)^{|n-3|}$ is linear phase with poles at $z = 2$ and $z = -2$, along with a triple pole at $z = 0$.
- Finite-duration sample responses with an even number of samples can also be linear phase, with the midpoint lying between two integers. For example, the sample response $h[n] = 1$ for $0 \le n \le 5$ and zero otherwise is linear phase. The midpoint of $h[n]$ is at $n = 2.5$ and hence the phase is $\angle H(e^{j\omega}) = -2.5\omega$. Systems like this which have an axis of symmetry that falls between two sample points are sometimes referred to as exhibiting *generalized linear phase*.

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