Carnegie Mellon ADSP Electrical & Computer ENGINEERING
Digital Signal Processing (18-491/691)
Fall Semester 2025
Department of Electrical and Computer Engineering

Partial Fraction Expansion for Multiple Poles in a Single Location

Introduction

Over the past two weeks we have looked into the solution of inverse z-transforms using partial fraction expansion for transfer functions $H(z)$ that are a ratio of two polynomials:

$$H(z) = \frac{\sum_{l=0}^{M} b_l z^{-l}}{\sum_{k=0}^{N} a_k z^{-k}}$$

The determination of the inverse z-transforms depends on the specifics of the form of $H(z)$.

Case I. $N > M$ and all poles are in unique locations

The first special case is when there are more poles than zeros (*i.e.* $N > M$), and that the poles are all in unique locations in the z-plane. In this case we can write directly

$$H(z) = \frac{\sum_{l=0}^{M} b_l z^{-l}}{\sum_{k=0}^{N} a_k z^{-k}} = \sum_{i=1}^{N} \frac{A_i}{1 - d_i z^{-1}}$$

In the expression above the d_i are the poles of the system and the A_i are the corresponding "*residues*," a term first introduced in the original formulation of the problem using complex calculus. The residues are easily determined using the relationship

$$A_k = H(z)\left(1 - d_i z^{-1}\right)\big|_{z=d_i}$$

Once $H(z)$ is in this form the solution can be obtained as a sum of exponentials because

$$\alpha^n u[n] \Leftrightarrow \frac{1}{1 - \alpha z^{-1}} \text{ with ROC } |z| > |\alpha| \text{ and}$$

$$-\alpha^n u[-n-1] \Leftrightarrow \frac{1}{1 - \alpha z^{-1}} \text{ with ROC } |z| < |\alpha|$$

The right-sided inverse transform will be used for poles that are inside the ROC and the left-sided inverse transform will be used for poles that are outside the ROC.

Case II. $M \geq N$ and all poles are in unique locations

As discussed in class, we use long division to reduce the "improper" fraction describing $H(z)$ to obtain a polynomial in z or in z^{-1} plus a remainder term that is in the form of a proper fraction (*i.e.* the order of the denominator is greater than the order of the numerator.) We arrange the numerator and denominator of $H(z)$ with powers of z that are increasingly negative when the solution is right-sided and with powers of z that are increasingly positive when the solution is left-sided. The remainder term, a proper fraction, is solved using the techniques of Case I.

The inverse of the polynomial in z or z^{-1} will be a series of discrete-time delta functions at a finite number of values of n and the remainder term, a proper fraction, is solved using the techniques of Case I.

Case III. Some multiple poles in the same location

The next case we will consider is that with at least some multiple poles in a single location. For now let us assume that we have s poles at $z = d_i$ and that the other poles of the system are all in unique locations. In this more general case we can represent the z-transform as

$$H(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{\substack{k=0 \\ k \neq i}}^{N} \frac{A_k}{1 - d_k z^{-1}} + \sum_{m=1}^{s} \frac{C_m}{(1 - d_i z^{-1})^m}$$

Note that the first sum represents the contribution of the terms for which $M \geq N$ as in Case II, the second term represents the contribution of the unique pole locations as in Case I, and the third sum represents the contribution of the single multiple pole.

As described in the text the coefficients C_m are obtained using the somewhat tedious expression

$$C_m = \frac{1}{(s-m)!(-d_i)^{s-m}} \left\{ \frac{d^{s-m}}{dw^{s-m}} (1 - d_i w)^s H(w^{-1}) \right\} \Big|_{w=d_i}^{-1}$$

Here the auxiliary variable w is a stand-in for z. The need for w arises from the fact that the pole and zero locations are defined in terms of z while the polynomials in $H(z)$ are expressed in terms of z^{-1}. The two equations above are Eqs. (3.46) and (3.47) in OSYP.

A simple example

Let us determine the unit sample response for a simple causal and stable system with a double pole at $z = 1/2$ and a single pole at $z = -1/4$. Let $H(z)$ be

$$H(z) = \frac{1}{\left(1 - \frac{1}{2}z^{-1}\right)^2 \left(1 + \frac{1}{4}z^{-1}\right)} \text{ with ROC } |z| > 1/2$$

From the expression above we write

$$H(z) = \frac{A_1}{1 + \frac{1}{4}z^{-1}} + \frac{C_1}{1 - \frac{1}{2}z^{-1}} + \frac{C_2}{\left(1 - \frac{1}{2}z^{-1}\right)^2}$$

Using the basic techniques of Case I of partial expansion, we can write

$$A_1 = H(z)\left(1+\frac{1}{4}z^{-1}\right)\bigg|_{z=-1/4} = \frac{1}{\left(1-\frac{1}{2}z^{-1}\right)^2}\bigg|_{z=-1/4} = \frac{1}{(1+2)^2} = \frac{1}{9}$$

Now setting $s=2$ and $m=1$ we obtain

$$C_1 = \frac{1}{1!\left(-\frac{1}{2}\right)^1}\frac{d}{dw}\left(1-\frac{1}{2}z^{-1}\right)^2 H(w^{-1}) = \frac{1}{1!\left(-\frac{1}{2}\right)^1}\frac{d}{dw}\left(1-\frac{1}{2}z^{-1}\right)^2 \frac{1}{\left(1+\frac{1}{4}w\right)\left(1-\frac{1}{2}w\right)^2}$$

$$C_1 = -2\left[\frac{-\frac{1}{4}}{\left(1+\frac{1}{4}w\right)^2}\right]\bigg|_{w=d^{-1}=2} = \frac{(-2)(-1/4)}{(3/2)^2} = \left(\frac{1}{2}\right)\left(\frac{4}{9}\right) = \frac{2}{9}$$

And now setting $s=2$ and $m=2$ we obtain

$$C_2 = \frac{1}{0!(-d_i)^0}\left(1-\frac{1}{2}w\right)^2 \frac{1}{\left(1-\frac{1}{2}w\right)^2\left(1+\frac{1}{4}w\right)}\bigg|_{w=2}$$

From these calculations we can write

$$H(z) = \frac{1/9}{\left(1+\frac{1}{4}z^{-1}\right)} + \frac{2/9}{\left(1-\frac{1}{2}z^{-1}\right)} + \frac{2/3}{\left(1-\frac{1}{2}z^{-1}\right)^2}$$

By inspection, given the ROC, the inverse z-transform (IZT) of $H(z)$ is

$$h[n] = \frac{1}{9}\left(-\frac{1}{4}\right)^n u[n] + \frac{2}{9}\left(\frac{1}{2}\right)^n u[n] \text{ plus the IZT of the third term of } H(z) \text{ above}$$

So the last thing we need to do is determin the IZT of

$$\frac{2/3}{\left(1-\frac{1}{2}z^{-1}\right)^2}$$

We can obtain this result by using the property that

$$nx[n] \Leftrightarrow -z\frac{dX(z)}{dz}$$

Applying this property to the right-sided decaying exponential we observe that for

$$\alpha^n u[n] \Leftrightarrow \frac{1}{1-\alpha z^{-1}}$$

$$n\alpha^n u[n] \Leftrightarrow -z\left[\frac{-1(-\alpha(-1)z^{-2}}{\left(1-\alpha z^{-1}\right)^2}\right]$$

$$n\alpha^n u[n] \Leftrightarrow +z\frac{\alpha z^{-2}}{\left(1-\alpha z^{-1}\right)^2} = \frac{z^{-1}\alpha}{\left(1-\alpha z^{-1}\right)^2}$$

Hence, using the time-shift property we observe that

$$(n+1)\frac{\alpha^{n+1}\alpha}{u}[n+1] = (n+1)\frac{\alpha^{n+1}\alpha}{u}[n] \Leftrightarrow \frac{1}{(1-\alpha z^{-1})^2}$$

Now we can obtain the complete expression

$$h[n] = \frac{1}{9}\left(-\frac{1}{4}\right)^n u[n] + \frac{2}{9}\left(\frac{1}{2}\right)^n u[n] + \frac{2}{3}(n+1)\left(\frac{1}{2}\right)^n u[n]$$

Collecting terms we can rewrite this result as

$$h[n] = \frac{1}{9}\left(-\frac{1}{4}\right)^n u[n] + \frac{8}{9}\left(\frac{1}{2}\right)^n u[n] + \frac{2}{3}n\left(\frac{1}{2}\right)^n u[n]$$

In general, having an s^{th}-order pole at a particular location in the z-plane (in this case $z = 1/2$) will cause the inverse transform for that exponential to be multiplied by a polynomial in n of order $s-1$. In this case the component of $h[n]$ that is $(1/2)^n$ is multiplied by the polynomial $(2/3)n + (8/9)$.