ZT INVERSES,

FREQUENCY RESPONSE + ZTs,

SPECIAL TYPES of LSI SYSTEMS

DSYP 3.3, 3.5, 5.0-5.3, 5.5-5.7

$$x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$\alpha^n u[n] \Leftrightarrow \frac{1}{1-\alpha z^{-1}}, \ |z| > |\alpha|$$

$$-\alpha^n u[-n-1] \Leftrightarrow \frac{1}{1-\alpha z^{-1}}, \ |z| < |\alpha|$$

IZT USING PARTIAL FRACTIONS

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{l=0}^{M} b_l z^{-l}}{\sum_{k=0}^{N} a_k z^{-k}}$$

CASE I: UNIQUE POLES, $M < N$

RESIDUES

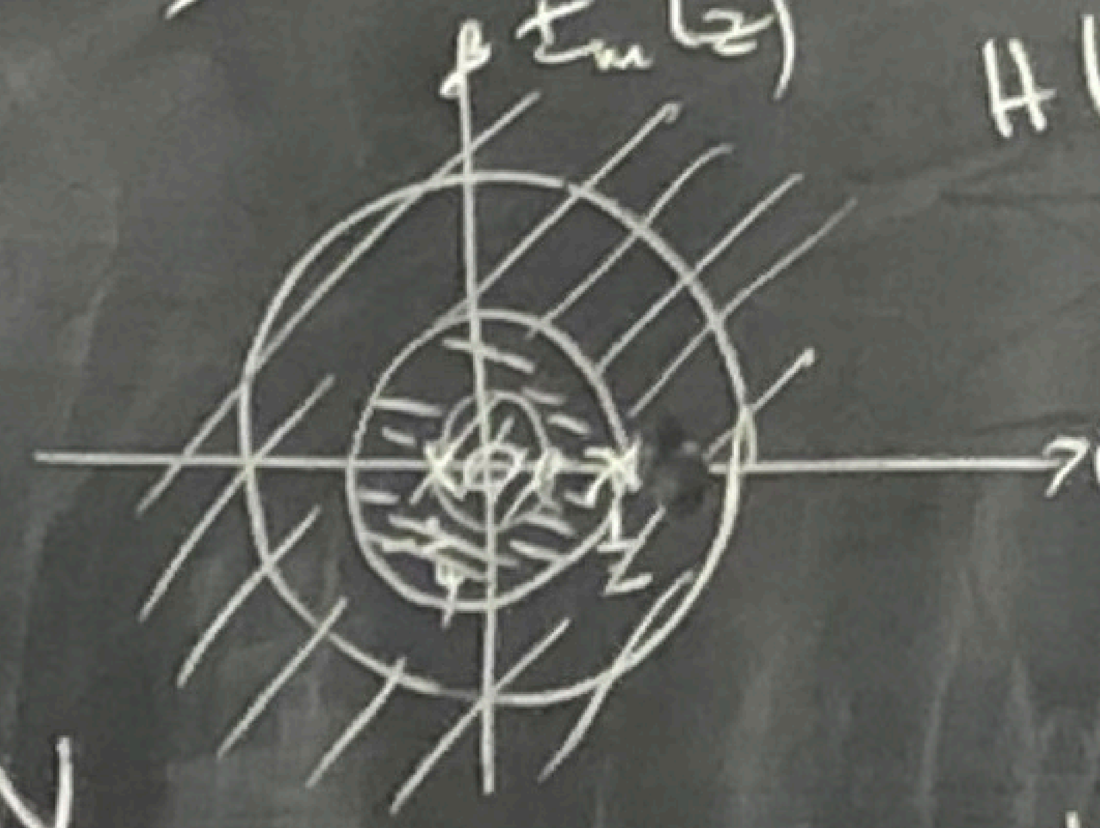
$$H(z) = \sum_{i=1}^{N} \frac{A_i}{1-z^{-1}d_i} \; ; \quad A_i = H(z)\left(1-z^{-1}d_i\right)\Big|_{z=d_i}$$

POLE LOC.

Ex.

$$H(z) = \frac{3-\frac{3}{4}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{4}z^{-1}\right)} = \frac{A_1}{1-\frac{1}{2}z^{-1}} + \frac{A_2}{1+\frac{1}{4}z^{-1}}$$

$A_1 = 1, \; A_2 = 2$



$$H(z) = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{2}{1+\frac{1}{4}z^{-1}}$$

ROC	$h[n]$
$\lvert z\rvert < \frac{1}{4}$	$\Rightarrow h[n] = -\left(\frac{1}{2}\right)^n u[-n-1] - 2\left(-\frac{1}{4}\right)^n u[-n-1]$
$\frac{1}{4} < \lvert z\rvert < \frac{1}{2}$	$\Rightarrow h[n] = -\left(\frac{1}{2}\right)^n u[-n-1] + 2\left(-\frac{1}{4}\right)^n u[n]$
$\lvert z\rvert > \frac{1}{2}$	$\Rightarrow h[n] = \left(\frac{1}{2}\right)^n u[n] + 2\left(-\frac{1}{4}\right)^n u[n]$

$$H(z) = \frac{3z\left(z-\frac{1}{4}\right)}{\left(z-\frac{1}{2}\right)\left(z+\frac{1}{4}\right)}$$

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$$x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} \, dz$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$\alpha^n u[n] \Leftrightarrow \frac{1}{1-\alpha z^{-1}}, \; |z| > |\alpha|$$

$$-\alpha^n u[-n-1] \Leftrightarrow \frac{1}{1-\alpha z^{-1}}, \; |z| < |\alpha|$$

PARTIAL FRACTIONS,

CASE II UNIQUE POLES, $M \geq N$

SOLUTION: REDUCE BY LONG DIVISION, PROCEED AS IN CASE I

Ex. $H(z) = \dfrac{3z^{-3} + 4z^{-2} + z^{-1} + 5}{1 - 3z^{-1}}$

$$\begin{array}{r}
-z^{-2} - \frac{5}{3}z^{-1} - \frac{8}{9} \\
-3z^{-1} + 1 \overline{\smash{\big)}\, 3z^{-3} + 4z^{-2} + z^{-1} + 5} \\
\underline{+3z^{-3} - z^{-2}} \\
5z^{-2} + z^{-1} \\
\underline{5z^{-2} - \frac{5}{3}z^{-1}} \\
\frac{8}{3}z^{-1} + 5 \\
\underline{\frac{8}{3}z^{-1} - 8/9} \\
53/9
\end{array}$$

$H(z) = -z^{-2} - \frac{5}{3}z^{-1} - \frac{8}{9} + \dfrac{53/9}{1 - 3z^{-1}}$

$|z| > 3$

$1 - 3z^{-1} = 0$

$z - 3 = 0 \Rightarrow z = 3$

$h[n] = -\delta[n-2] - \frac{5}{3}\delta[n-1] - \frac{8}{9}\delta[n] + \frac{53}{9}(3)^n u[n]$

PARTIAL FRACTION CASE III.

MULTIPLE POLES IN SAME LOCATION

SUPPOSE WE HAVE s MULTIPLE POLES IN LOCATION d_i

$$H(z) = \frac{\sum_{\ell=0}^{M} b_\ell z^{-\ell}}{\sum_{k=0}^{N} a_k z^{-k}} = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{\substack{k=1 \\ k\neq i}}^{N} \frac{A_k}{1-d_k z^{-1}} + \sum_{m=1}^{s} \frac{C_m}{(1-d_i z^{-1})^m}$$

$w = z^{-1}$

$$C_m = \frac{1}{(s-m)!\,(-d_i)^{s-m}} \cdot \left\{ \frac{d^{s-m}}{dw^{s-m}} (1-d_i w)^s H(w^{-1}) \right\}\Big|_{w=d_i^{-1}}$$

Ex. $$H(z) = \frac{1}{(1+\frac{1}{4}z^{-1})(1-\frac{1}{2}z^{-1})^2} = \frac{A_1}{1+\frac{1}{4}z^{-1}} + \frac{C_1}{(1-\frac{1}{2}z^{-1})} + \frac{C_2}{(1-\frac{1}{2}z^{-1})^2}$$

$$A_1 = H(z)\left(1+\tfrac{1}{4}z^{-1}\right)\Big|_{z=-\frac{1}{4}} = \frac{1}{(1-\frac{1}{2}z^{-1})^2}\Big|_{z=-\frac{1}{4}} = \frac{1}{(1+2)^2} = \frac{1}{9}$$

$d_i = \frac{1}{2} \Rightarrow w = 2$

$$C_1 = \frac{1}{1!\,(-\frac{1}{2})} \cdot \frac{d}{dw} \cancel{(1-\tfrac{1}{2}w)^2} \frac{1}{(1+\frac{1}{4}w)\cancel{(1-\frac{1}{2}w)^2}}\Big|_{w=2}$$

$$\frac{-\frac{1}{4}}{(1+\frac{1}{4}w)^2}\Big|_{w=2} = \frac{-1/4}{9/4} = -1/9$$

$$C_1 = \frac{2}{9}$$

$s = 2, m = 2$

$$C_2 = \frac{1}{(2-2)!\,(-d_1)^0} \frac{d^0}{dw^0} \frac{\cancel{(1-\frac{1}{2}w)^2}}{(1+\frac{1}{4}w)\cancel{(1-\frac{1}{2}w)^2}}\Bigg|_{w=2} = \frac{1}{1+\frac{1}{4}\cdot 2} = \frac{2}{3}$$

SUPPOSE ROC IS $|z| > 1/2$

$$H(z) = \frac{1/9}{1+\frac{1}{4}z^{-1}} + \frac{2/9}{1-\frac{1}{2}z^{-1}} + \frac{2/3}{(1-\frac{1}{2}z^{-1})^2}$$

$$h[n] = \frac{1}{9}\left(-\frac{1}{4}\right)^n u[n] + \frac{2}{9}\left(\frac{1}{2}\right)^n u[n] + \frac{2}{3}(n+1)\left(\frac{1}{2}\right)^n u[n]$$

$$\frac{8}{9}\left(\frac{1}{2}\right)^n u[n] + \frac{2}{3} n \left(\frac{1}{2}\right)^n u[n]$$

$$\alpha^n u[n] \Leftrightarrow \frac{1}{1-\alpha z^{-1}}, \quad |z| > \alpha$$

$$n\, x[n] \Leftrightarrow -z \frac{d X(z)}{dz}$$

$$n\, \alpha^n u[n] \Leftrightarrow -z \frac{d}{dz}\left(\frac{1}{1-\alpha z^{-1}}\right) = -z\left[\frac{+\alpha z^{-2}}{(1-\alpha z^{-1})^2}\right] = \frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$$

$$n\, \alpha^{n-1} u[n] \Leftrightarrow \frac{\cancel{\alpha} z}{(1-\alpha z^{-1})^2}$$

$$(n+1)\alpha^n u[n] \; \cancel{u[n+1]} \Leftrightarrow \frac{1}{(1-\alpha z^{-1})^2} \quad |z| > |\alpha|$$

RELATING POLE + ZERO LOCATIONS TO THE MAGNITUDE + PHASE OF THE DTFT

$$H(z) = \frac{\sum_{\ell=0}^{M} b_\ell z^{-\ell}}{\sum_{k=0}^{N} a_k z^{-k}} = \frac{b_0}{a_0} \frac{\prod_{\ell=1}^{M} (1 - c_\ell z^{-1})}{\prod_{k=1}^{N} (1 - d_k z^{-1})} = z^{N-M} \frac{b_0}{a_0} \frac{\prod_{\ell=1}^{M} (z - c_\ell)}{\prod_{k=1}^{N} (z - d_k)}$$

$$H(e^{j\omega}) = H(z)\Big|_{z=e^{j\omega}}$$

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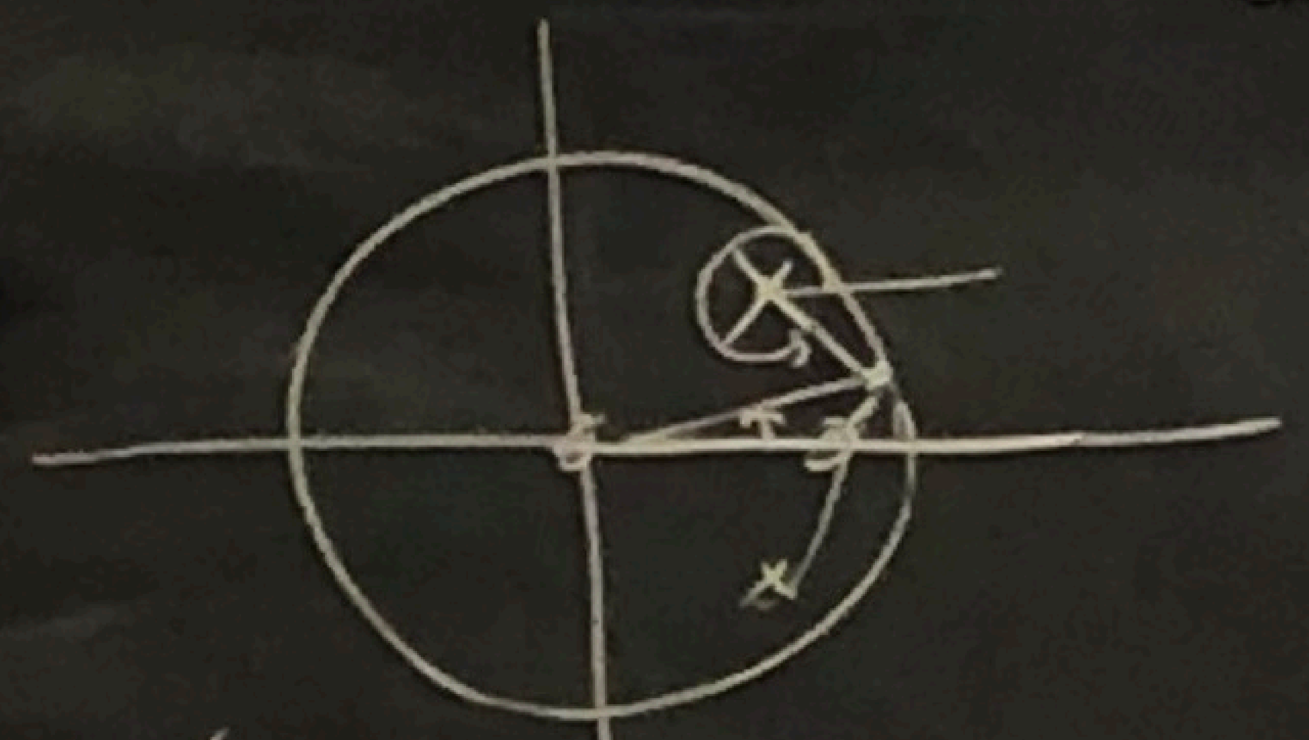
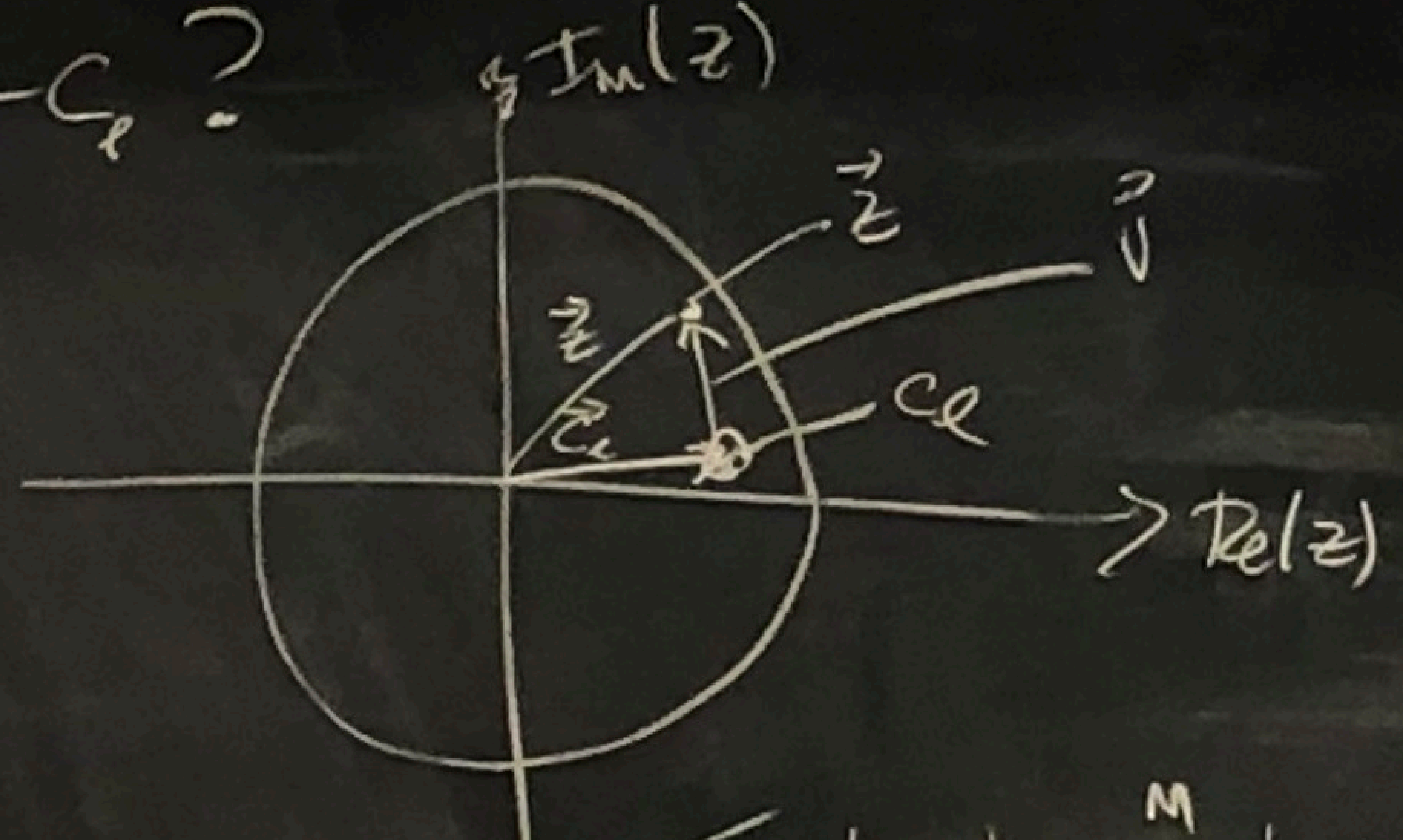
WHAT IS $z - c_\ell$?





$\vec{c}_\ell + \vec{v} = \vec{z}$

$\vec{z} - \vec{c}_\ell = \vec{v}$

DTFT is $H(e^{j\omega}) = H(z)\Big|_{z=e^{j\omega}}$

$$|H(z)| = |z^{N-M}| \left|\frac{b_0}{a_0}\right| \frac{\prod_{\ell=1}^{M} |z - c_\ell|}{\prod_{k=1}^{N} |z - d_k|}$$

$$\angle H(z) = \angle(z^{N-M}) + \angle\left(\frac{b_0}{a_0}\right) + \sum_{\ell=1}^{M} \angle(z - c_\ell) - \sum_{k=1}^{N} \angle(z - d_k)$$

$$|H(e^{j\omega})| = H(z)\Big|_{z=e^{j\omega}} = \cancel{|e^{j\omega(N-M)}|} \cdot \left|\frac{b_0}{a_0}\right| \frac{\prod_{\ell=1}^{M} |e^{j\omega} - c_\ell|}{\prod_{k=1}^{N} |e^{j\omega} - d_k|}$$

$$\angle H(e^{j\omega}) = \angle H(z)\Big|_{z=e^{j\omega}} = \underbrace{\angle e^{j\omega(N-M)}}_{\omega(N-M)} + \underbrace{\angle\left(\frac{b_0}{a_0}\right)}_{\{0,\pi\}} + \sum_{\ell=1}^{M} \angle(e^{j\omega} - c_\ell) - \sum_{k=1}^{N} \angle(e^{j\omega} - d_k)$$