Carnegie Mellon DSP Electrical & Computer ENGINEERING

Fundamentals of Signal Processing (18-491)
Spring Semester, 2019

PROPERTIES AND INVERSES OF *Z*-TRANSFORMS

Note: These notes originally accompanied a video lecture on z-transforms some years ago. They are provided this year as a complementary resource to the text and the class notes.

I. Introduction

In the last lecture we reviewed the basic properties of the z-transform and the corresponding region of convergence. In this lecture we will cover

- Stability and causality and the ROC of the z-transform (see Lecture 6 notes)
- Comparison of ROCs of z-transforms and LaPlace transforms (see Lecture 6 notes)
- Basic z-transform properties
- Linear constant-coefficient difference equations and z-transforms
- Evaluation of the inverse z-transform using
    - Direct evaluation (not done in detail in this course)
    - Partial fraction evaluation
    - Evaluation using long division
    - Evaluation using Taylor series

II. Basic *z*-transform properties

Introduction: While the basic *z*-transform properties are very similar to those of the corresponding DTFTs, they are complicated a little by the fact that we now must also consider the region of convergence of the new transform as well.

We will not review right now all of the properties cited in the text, but we will touch on the most important ones.

If the time function $x[n]$ has the *z*-transform $X(z)$, with the corresponding ROC R_x, we observe the following general properties of functions and their *z*-transforms:

- **Linearity**

$$ax_1[n] + bx_2[n] \Leftrightarrow aX_1(z) + bX_2(z)$$

with the ROC being the "overlap" region of the ROCs R_{x_1} and R_{x_2} or $R_{x_1} \cap R_{x_2}$

- **Time shift**

$$x[n-N] \Leftrightarrow z^{-N}X(z) \text{ with ROC } R_x \text{ (although possibly excluding } z = 0\text{)}$$

This relation plays a big role in dealing with difference equations, as will be discussed below.

- **Multiplication in time by a complex exponential**

$$a^n x[n] \Leftrightarrow X\left(\frac{1}{a}\right) \text{ with ROC } |a|R_x$$

Note that if *a* is purely real, this corresponds to a circularly-symmetric *expansion or contraction* of the *z*-plane. If *a* is purely imaginary, this corresponds to a *rotation* of the *z*-plane.

- **Convolution in time**

$$x[n]*h[n] \Leftrightarrow X(z)H(z) \text{ with ROC } R_x \cap R_h$$

This, of course, is just like things are with DTFTs.

- **General multiplication in time**

$$x[n]w[n] \Leftrightarrow \frac{1}{2\pi j}\oint X(\nu)W\left(\frac{z}{\nu}\right)\nu^{-1}d\nu$$

This is again a contour integral, which we cannot evaluate using the techniques developed in this class. Hence, we will normally use DTFTs rather than *z*-transforms whenever we need to consider the *z*-transforms of multiplied time functions. We will, however, make use of the relationship that describes the *z*-transforms of a function multiplied by a complex exponential.

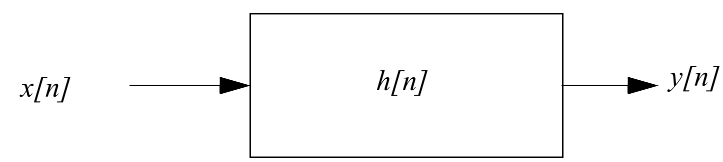
- **Differentiation of the *Z*-transform**

$$nx[n] \Leftrightarrow -z\frac{dX(z)}{dz}$$

This is easily proved by differentiating the *z*-transform equation with respect to *z.* It plays an important role in dealing with systems where multiple poles occur in the same location n the *z*-plane.

III. Linear constant-coefficient difference equations

Introduction:



- Difference equations in discrete time play the same role in characterizing the time-domain response of discrete-time LSI systems that differential equations play for continuous-time LTI systems.

- In most general form we can write difference equations as

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]$$

where (as usual) $x[n]$ represents the input and $y[n]$ represents the output. Since we can set a_0 equal to 0 without any loss of generality, we can rewrite this as

$$y[n] = -\sum_{k=1}^{N} a_k y[n-k] + \sum_{m=0}^{M} b_m x[n-m]$$

Comments:

- In this representation we characterize the present output of an LSI system as a linear combination of past outputs combined with a linear combination of the present and previous inputs.

- The difference equations alone do not uniquely specify the system. Initial conditions are needed as well. Normally we assume initial rest (*i.e.* the output is zero before the input is applied). Otherwise the system would be neither linear nor shift-invariant, as discussed in class.

- These equations can be solved analytically, just as in the case of ordinary differential equations. Normally the solution involves obtaining the homogenous solution (or the natural frequencies) of the system, and the particular solution (or the forced response). Details and examples of this are provided in the text.

We can solve these equations using the procedure of iteration. For example, consider the equation

$$y[n] - \frac{1}{4}y[n-1] - \frac{1}{8}y[n-2] = 3x[n] - \frac{3}{4}x[n-1]$$

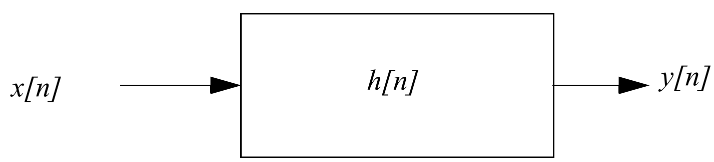
or,

$$y[n] = \frac{1}{4}y[n-1] + \frac{1}{8}y[n-2] + 3x[n] - \frac{3}{4}x[n-1]$$

Let’s obtain the unit sample response $h[n]$ for this equation via iteration. As we did in class a few days ago, this can be done by setting up the table:

n	*x[n-1]*	*x[n]*	*y[n-2]*	*y[n-1]*	*y[n]*
0	0	1	0	0	3
1	1	0	0	3	0
2	0	0	3	0	3/8

IV. Poles and zeros of LSI systems



Let’s consider again the generic LSI system with $y[n] = x[n]*h[n]$ or $Y(z) = X(z)H(z)$.

You will recall that the general difference equation for such a system is

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]$$

Taking the *z*-transforms of both sides, we obtain

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{m=0}^{M} b_m z^{-m} X(z)$$

$$Y(z) \sum_{k=0}^{N} a_k z^{-k} = X(z) \sum_{m=0}^{M} b_m z^{-m}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{m=0}^{M} b_m z^{-m}}{\sum_{k=0}^{N} a_k z^{-k}}$$

In other words, if we have an LSI system that is characterizable by a linear constant-coefficient difference equation, the z-transform of the unit sample response, which we refer to as the *system function*, will always be the ratio of two polynomials in z^{-1}, with coefficients that are the coefficients of the corresponding difference equation. Virtually all of the systems that we will encounter in this course will be of this form.

For example, in the case of the difference equation we had looked at previously,

$$y[n] - \frac{1}{4}y[n-1] - \frac{1}{8}y[n-2] = 3x[n] - \frac{3}{4}x[n-1]$$

we obtain

$$H(z) = \frac{Y(z)}{X(z)} = \frac{3 - \frac{3}{4}z^{-1}}{1 - \frac{1}{4}z^{-1} - \frac{1}{8}z^{-2}}$$

Now, the *zeros* of the systems are by definition the values of z that cause the numerator of $H(z)$ to equal zero (*i.e.* the roots of the numerator polynomial in z). Similarly, the *poles* of the system are the values of z that cause the denominator of $H(z)$ to go to zero (or the roots of the denominator polynomial in z). We can obtain the poles and zeros of our example by multiplying numerator and denominator by z^2.

$$H(z) = \frac{z^2\left(3 - \frac{3}{4}z^{-1}\right)}{z^2\left(1 - \frac{1}{4}z^{-1} - \frac{1}{8}z^{-2}\right)} = \frac{3z^2 - \frac{3}{4}z}{z^2 - \frac{1}{4}z - \frac{1}{8}} = \frac{3z\left(z - \frac{1}{4}\right)}{\left(z - \frac{1}{2}\right)\left(z + \frac{1}{4}\right)}$$

So, in this system, the zeros are at $z = 0$ and $z = \frac{1}{4}$ and the poles are at $z = \frac{1}{2}$ and $z = -\frac{1}{4}$.

Pole locations and the ROC

We note that since the z-transform is infinite at the values of z corresponding to the pole locations, the ROC cannot include the locations of the system's poles. In fact, ROCs are always bounded by circles that are centered at the origin of the z-plane, and that pass through the locations of the poles. In this case, the potential boundaries of the ROCs are circles of radius 1/4 and 1/2. This means that there are three possible ROCs for this system:

- $|z| < \frac{1}{4}$ This system would be unstable and have a left-sided unit sample response.

- $\frac{1}{4} < |z| < \frac{1}{2}$ This system would also be unstable and have a "both-sided" unit sample response.

- $|z| > \frac{1}{2}$ This system would be stable and have a right-sided ("causal") unit sample response.

The exact ROC would be known either because it would be given or because you will know whether the system is causal and/or stable.

V. Inverse *z*-transforms

Recall that the equations that define *z*-transforms are

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

$$x[n] = \frac{1}{2\pi j}\oint_c X(z)z^{n-1}dz$$

As we know, we will not evaluate the complex contour integral for the inverse *z*-transform directly. Instead we will use one of the following techniques:

- Partial fraction expansion
- Long division
- Taylor series expansion

Partial fraction expansion

As is discussed briefly in the videotape, partial fraction expansion is a computational hack algorithm that enables us to obtain the results that we would have obtained if we had gone through the formal procedure of contour integration over the complex *z*-plane.

The partial fraction method of obtaining inverse *z*-transforms builds on the fact that we know that

$$a^n u[n] \Leftrightarrow \frac{z}{z-a} = \frac{1}{1-az^{-1}} \text{ for the ROC } |z| > |a| \text{ and that}$$

$$-a^n u[-n-1] \Leftrightarrow \frac{1}{1-az^{-1}} \text{ for the ROC } |z| < |a|$$

The simplest case:

If

1. the order of the numerator of the polynomial in z^{-1} is less than the order of its denominator (as it is

in this case), and

2. all the poles of the z-transform are at different locations in the z-plane (as they are in this case),

then we can write for the example we have been considering

$$H(z) = \frac{3 - \frac{3}{4}z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 + \frac{1}{4}z^{-1}\right)} = \frac{A_1}{\left(1 - \frac{1}{2}z^{-1}\right)} + \frac{A_2}{\left(1 + \frac{1}{4}z^{-1}\right)}$$

where the as-yet undetermined coefficients are referred to as the *residues* of the z-transform, following the term used in complex calculus.

In general, the residues A_i corresponding to a pole at location $z = z_i$ can be easily obtained using the formula

$$A_i = H(z)(z - z^i)\Big|_{z = z_i}$$

For our example this becomes:

$$A_1 = H(z)\left(1 - \frac{1}{2}z^{-1}\right)\Bigg|_{z = \frac{1}{2}} = \left.\frac{3 - \frac{3}{4}z^{-1}}{1 + \frac{1}{4}z^{-1}}\right|_{z = \frac{1}{2}} = \frac{\frac{3}{2}}{\frac{3}{2}} = 1$$

$$A_2 = H(z)\left(1 + \frac{1}{4}z^{-1}\right)\Bigg|_{z = -\frac{1}{4}} = \left.\frac{3 - \frac{3}{4}z^{-1}}{1 + -\frac{1}{2}z^{-1}}\right|_{z = -\frac{1}{4}} = \frac{6}{3} = 2$$

Hence

$$H(z) = \frac{1}{\left(1 - \frac{1}{2}z^{-1}\right)} + \frac{2}{\left(1 + \frac{1}{4}z^{-1}\right)}$$

and if we are told that the system is causal, then the corresponding inverse z-transform is

$$h[n] = \left[\left(\frac{1}{2}^n\right) + 2\left(\frac{-1}{4}\right)^n\right]u[n]$$

Note that the first samples of this function starting with $n = 0$ are 3, 0, and 3/8, as determined previously by iteration.

There are actually three possible inverse z-transforms transforms for this system function:

1. If the ROC is $|z| > \frac{1}{2}$, $h[n] = \left[\left(\frac{1}{2}\right)^n + 2\left(\frac{-1}{4}\right)^n\right]u[n]$, as noted above. This system is causal and stable.

2. If the ROC is $\frac{1}{4} < |z| < \frac{1}{2}$, $h[n] = -\left(\frac{1}{2}\right)^n u[-n-1] + 2\left(\frac{-1}{4}\right)^n u[n]$. This system is neither causal nor stable.

3. If the ROC is $|z| < \frac{1}{4}$, $h[n] = \left[-\left(\frac{1}{2}\right)^n u[-n-1] + \left(\frac{-1}{4}\right)^n\right]u[-n-1]$. This system is also neither causal nor stable.

Partial fractions with numerator order greater than or equal to denominator order:

If the order of the numerator is too large, we can reduce it via long division. For example, if we have the transform

$$H(z) = \frac{3z^{-3} + 4z^{-2} + z^{-1} + 5}{1 - 3z^{-1}}$$, we can apply long division as follows:

(Using MATLAB notation and setting in plain text to control positioning:)

```
                   -z^(-2)-(5/3)z^(-1)-(8/9)
                  _____________________________
-3z^(-1)+1  )3z^(-3)+4z^(-2)+     z^(-1)+ 5
             3z^(-3)- z^(-2)
                   5z^(-2)+     z^(-1)
                   5z^(-2)-(5/3)z^(-1)
                           (8/3)z^(-1)+ 5
                           (8/3)z^(-1)-(8/9)
                                      53/9
```

In other words, the result of this division operation is that

$$H(z) = \frac{3z^{-3} + 4z^{-2} + z^{-1} + 5}{1 - 3z^{-1}} = -z^{-2} - \frac{5}{3}z^{-1} - \frac{8}{9} + \frac{53/9}{1 - 3z^{-1}}$$

and the corresponding inverse z-transform equals $-\delta[n-2] - \frac{5}{3}\delta[n-1] - \frac{8}{9}\delta[n]$ plus the inverse z-transform of $\frac{53/9}{1 - 3z^{-1}}$, which now has a denominator of higher order than the numerator, and which would be either $(53/9)(3)^n u[n]$ or $-(53/9)(3)^n u[-n-1]$, depending on whether the ROC is $|z| > 3$ or $|z| < 3$, respectively.

Partial fractions with multiple poles in the same location:

If we have multiple poles in the same location, the situation is more difficult. As described in the text, If we have s poles at location d in the z-plane, the contribution of the multiple pole to the partial-fraction of the z-transform is

$$\sum_{m=1}^{s} \frac{C_m}{(1-dz^{-1})^m}$$

where

$$C_m = \frac{1}{(s-m)!(-d)^{s-m}} \left\{ \frac{d^{s-m}}{dw^{s-m}} (1-dw)^s H(w^{-1}) \right\} \text{ evaluated at } w = d^{-1}$$

While the use of $w = z^{-1}$ in the equation above may seem needlessly confusing and arbitrary, it actually is helpful because the derivatives are easier to evaluate in terms of w than in terms of z^{-1}.

In the general case, with some isolated poles, one set of multiple poles in the same location, and a polynomial of numerator order M and denominator order N, the z-transform would be of the form

$$H(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1,\, k \neq i}^{N} \frac{A_k}{1-d_k z^{-1}} + \sum_{m=1}^{s} C_m (1-d_i z^{-1})^m$$

where d_i is the location of the pole of multiplicity s. If there is more than one location with multiple poles, additional series of terms similar to the latter term above will be obtained.

Let us consider a simple example with multiple poles:

$$H(z) = \frac{1}{\left(1-\frac{1}{2}z^{-1}\right)^2 \left(1+\frac{1}{4}z^{-1}\right)}$$

This transfer function, of course, has a single pole at $z = -1/4$ and a double pole at $z = 1/2$.

Using partial fractions, we would like to rewrite this equation in the form of

$$H(z) = \frac{A_1}{\left(1+\frac{1}{4}z^{-1}\right)} + \frac{C_1}{\left(1-\frac{1}{2}z^{-1}\right)} + \frac{C_2}{\left(1-\frac{1}{2}z^{-1}\right)^2}$$

The residue A_1 is obtained in conventional fashion:

$$A_1 = H(z)\left(1+\frac{1}{4}z^{-1}\right)\Bigg|_{z=-\frac{1}{4}} = \frac{1}{\left(1-\frac{1}{2}z^{-1}\right)^2}\Bigg|_{z=-\frac{1}{4}} = \frac{1}{3^2} = \frac{1}{9}$$

For the residue C_1 we will solve the equation

$$C_m = \frac{1}{(s-m)!(-d)^{s-m}}\left\{\frac{d^{s-m}}{dw^{s-m}}(1-dw)^s H(w^{-1})\right\}\Bigg|_{w=d^{-1}}$$

for $s = 2$, $m = 1$, $w = z^{-1}$, $w^{-1} = z$, and $d = 1/2$. This produces

$$C_1 = \frac{1}{(1)!(-1/2)^1}\left\{\frac{d^1}{dw^1}\left(1-\frac{1}{2}w\right)^2\frac{1}{\left(1-\frac{1}{2}w\right)^2\left(1+\frac{1}{4}w\right)}\right\}\Bigg|_{w=2} = -2\left[\frac{-1/4}{\left(1+\frac{1}{2}\right)^2}\right] = \frac{2}{9}$$

For the residue C_2 we solve the same equation but with $m = 2$. This produces

$$C_2 = \frac{1}{(0)!(-1/2)^0}\left\{\frac{d^0}{dw^0}\left(1-\frac{1}{2}w\right)^2\frac{1}{\left(1-\frac{1}{2}w\right)^2\left(1+\frac{1}{4}w\right)}\right\}\Bigg|_{w=2} = \frac{1}{1+\frac{2}{4}} = \frac{2}{3}$$

Combining the terms, we obtain

$$H(z) = \frac{1/9}{1+\frac{1}{4}z^{-1}} + \frac{2/9}{1-\frac{1}{2}z^{-1}} + \frac{2/3}{\left(1-\frac{1}{2}z^{-1}\right)^2}$$

You can verify that this is the correct partial-fraction expansion by multiplying the terms back together or through the use of the MATLAB command `residuez`.

As noted above, we can obtain the inverse z-transform for the final term through the use of the differentiation property for z-transforms:

$$nx[n] \Leftrightarrow -z\frac{dX(z)}{dz}$$

Applying this property to the case of the decaying exponential signal produces the transform pairs

$$na^n u[n] \Leftrightarrow \frac{az^{-1}}{1-az^{-1}}$$ for the ROC $|z| > |a|$ and

$$-na^n u[-n-1] \Leftrightarrow \frac{az^{-1}}{1-az^{-1}} \text{ for the ROC } |z| < |a|$$

Hence, for an ROC of $|z| > |a|$ we can use the shift property of the z-transform to obtain

$$a^{-1}(n+1)a^{n+1}u[n+1] \Leftrightarrow \frac{1}{(1-az^{-1})^2}$$

Because $n+1 = 0$ for $n = -1$, we can rewrite this expression a little more simply as

$$(n+1)a^n u[n] \Leftrightarrow \frac{1}{(1-az^{-1})^2} \text{ for } |z| > |a|$$

Note that higher-order poles would yield components of the corresponding inverse *z*-transforms that include exponentials multiplied by polynomials of a higher order, with the order of the polynomial equal to the multiplicity of the pole at a given location minus 1.

If the ROC of the *z*-transform in question is $|z| > 1/2$, the inverse transform can be obtained by combining all of the inverse transforms:

$$\left[\left(\frac{1}{9}\right)\left(-\frac{1}{4}\right)^n + \left(\frac{2}{9}\right)\left(\frac{1}{2}\right)^n + \left(\frac{2}{3}\right)(n+1)\left(\frac{1}{2}\right)^n\right]u[n] \Leftrightarrow \frac{1/9}{\left(1+\frac{1}{4}z^{-1}\right)} + \frac{2/9}{\left(1-\frac{1}{2}z^{-1}\right)} + \frac{2/3}{\left(1-\frac{1}{2}z^{-1}\right)^2}$$

Inverse *z*-transforms by long division

Long division can also be used to obtain inverse *z*-transforms numerically. For example, consider again the transform

$$H(z) = \frac{3-\frac{3}{4}z^{-1}}{1-\frac{1}{4}z^{-1}+\frac{1}{8}z^{-2}}$$

Arranging the terms in order of increasing powers of z^{-1} and dividing we obtain

```
                                    3+ 0z^(-1)+(3/8)z^(-2)+ .
1-(1/4)z^(-1)-(1/8)z^(-2) ) 3-(3/4)z^(-1)+  0z^(-2)+ 0z^(-3)+ 0z^(-4)+ ...
                            3-(3/4)z^(-1)-(3/8)z^(-2)
                                   0z^(-1)+(3/8)z^(-2)+ 0z^(-3)
                                   0z^(-1) - 0z^(-2) + 0 z^(-3)
                                            (3/8)z^(-2)+0 z^(-3)+ 0z^(-4)
```

Hence, the first several terms of the quotient will be

$$H(z) = 3 + 0z^{-1} + \frac{3}{8}z^{-2} + \ldots$$

which, if causal, has by inspection the inverse transform of

$$h[n] = 3\delta[n] + 0\delta[n-1] + \frac{3}{8}\delta[n-2] + \ldots$$

Comment:

- A left-sided inverse *z*-transform could be obtained in similar fashion by arranging the terms in the opposite order. This technique cannot be used to obtain both-sided inverse *z*-transforms.

Inverse *z*-transforms by Taylor series expansion

Occasionally we are asked to obtain the inverse *z*-transform of a function that is not a ratio of polynomials in z or z^{-1}. Sometimes we can use Taylor series expansion to accomplish this.

For example, the *z*-transform $X(z) = \ln\left(\frac{1+z^{-1}}{1-z^{-1}}\right)$ with ROC $|z| > 1$ can be obtained using series expansion:

$$X(z) = \ln\left(\frac{1+z^{-1}}{1-z^{-1}}\right) = 2\left[z^{-1} + \frac{z^{-3}}{3} + \frac{z^{-5}}{5} + \frac{z^{-7}}{7} + \ldots\right]$$

The corresponding inverse is obtained easily by inspection as in the case of the results of long division.