2/5/24 Z-TRANSFORM INVERSES

(O&S 4ED 3.3, 3.5)

ZT $\quad X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$

IZT $\quad x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz$

APPROACHES TO THE IZT:

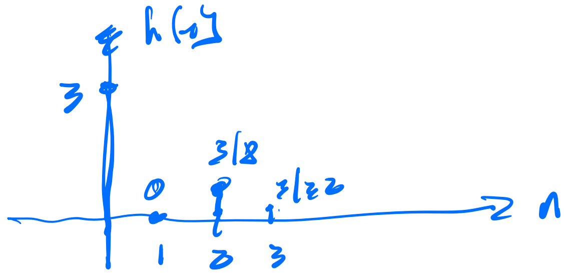
1. DIRECT COMPUTATION USING CONTOUR INTEGRATION
2. ✓ LONG DIVISION
3. ★ PARTIAL FRACTION EXPANSION
4. TAYLOR SERIES EXPANSION
5. ✓ ITERATION

CONSIDER LSI SYSTEMS FOR WHICH

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{l=0}^{M} b_l z^{-l}}{\sum_{k=0}^{N} a_k z^{-k}} = \frac{G \sum_{l=0}^{M} b_l z^{-l}}{\prod_{k=1}^{N} (1 - d_k z^{-1})}$$

★ $y[n] = \frac{1}{4} y[n-1] + \frac{1}{8} y[n-2] + 3x[n] - \frac{3}{4} x[n-1]$

$$H(z) = \frac{3 - \frac{3}{4} z^{-1}}{(1 - \frac{1}{2} z^{-1})(1 + \frac{1}{4} z^{-1})} = \frac{3 - \frac{3}{4} z^{-1}}{1 - \frac{1}{4} z^{-1} - \frac{1}{8} z^{-2}} = \frac{Y(z)}{X(z)}$$



SOLVING FOR IZT USING PARTIAL FRACTIONS

WE KNOW

$$\alpha^n u[n] \Longleftrightarrow \frac{1}{1-\alpha z^{-1}}, \quad |z| > |\alpha|$$

$$-\alpha^n u[-n-1] \Longleftrightarrow \frac{1}{1-\alpha z^{-1}}, \quad |z| < |\alpha|$$

HERE $H(z) = \dfrac{3 - \frac{3}{4} z^{-1}}{(1-\frac{1}{2}z^{-1})(1+\frac{1}{4}z^{-1})} = \dfrac{A_1}{1-\frac{1}{2}z^{-1}} + \dfrac{A_2}{1+\frac{1}{4}z^{-1}}$

CASES:

1. $M < N$, $\forall$ POLES IN UNIQUE LOCATIONS
2. $M \geq N$, $\forall$ POLES IN UNIQUE LOCATIONS
3. $M < N$, MULTIPLE POLES FOR AT LEAST ONE LOCATION IN Z-PLANE

CASE I, $M < N$ UNIQUE POLE LOCATIONS

$$H(z) = \frac{B(z)}{A(z)} = \sum_{k=1}^{N} \frac{A_k}{1-d_k z^{-1}}$$

$$A_k = H(z)(1-d_k z^{-1})\Big|_{z=d_k}$$

$$H(z) = \frac{3 - \frac{3}{4} z^{-1}}{(1-\frac{1}{2}z^{-1})(1+\frac{1}{4}z^{-1})} = \frac{A_1}{1-\frac{1}{2}z^{-1}} + \frac{A_2}{1+\frac{1}{4}z^{-1}}$$

$$A_1 = \frac{(3-\frac{3}{4}z^{-1})\cancel{(1-\frac{1}{2}z^{-1})}}{\cancel{(1-\frac{1}{2}z^{-1})}(1+\frac{1}{4}z^{-1})}\Bigg|_{z=\frac{1}{2}} = \frac{3 - \frac{3}{4}\cdot 2}{1+\frac{1}{4}2} = \frac{3/2}{3/2} = 1$$

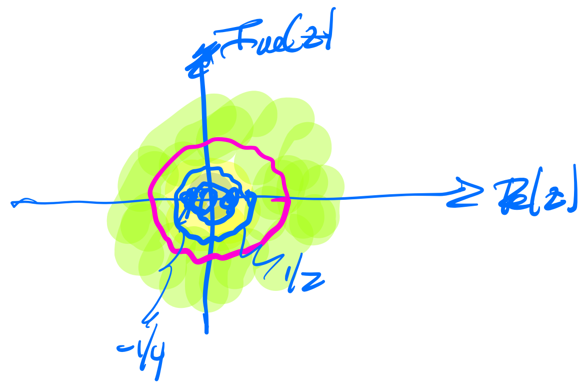
$$A_z = \left.\frac{\left(3-\frac{3}{4}z^{-1}\right)\cancel{\left(1+\frac{1}{4}z^{-1}\right)}}{\left(1-\frac{1}{2}z^{-1}\right)\cancel{\left(1+\frac{1}{4}z^{-1}\right)}}\right|_{z=-1/4} = \frac{3+\frac{3}{4}4}{1+\frac{1}{2}\cdot 4} = \frac{6}{3} = 2$$

$$H(z) = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{2}{1+\frac{1}{4}z^{-1}} = \frac{1+\frac{1}{4}z^{-1}+2-z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{4}z^{-1}\right)} = \frac{3-\frac{3}{4}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{4}z^{-1}\right)} \quad *$$

$$H(z) = \frac{3-\frac{3}{4}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{4}z^{-1}\right)} = \frac{z\left(3z-\frac{3}{4}\right)}{\left(z-\frac{1}{2}\right)\left(z+\frac{1}{4}\right)} = \frac{3z\left(z-\frac{1}{4}\right)}{\left(z-\frac{1}{2}\right)\left(z+\frac{1}{4}\right)}$$

ZEROS @ $0, 1/4$

POLES @ $1/2, -1/4$



POSSIBLE ROCs...

- $|z| > \frac{1}{2}$
- $|z| < \frac{1}{4}$
- $\frac{1}{4} < |z| < \frac{1}{2}$

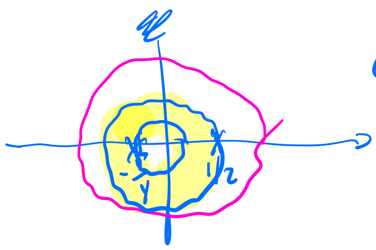
$$\alpha^n u[n] \Longleftrightarrow \frac{1}{1-\alpha z^{-1}}, \quad |z| > |\alpha|$$

$$-\alpha^n u[-n-1] \Longleftrightarrow \frac{1}{1-\alpha z^{-1}}, \quad |z| < |\alpha|$$

INVERSE TRANSFORMS

$$H(z) = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{2}{1+\frac{1}{4}z^{-1}}$$

- $|z| > \frac{1}{2} \rightarrow h[n] = \left(\frac{1}{2}\right)^n u[n] + 2\left(-\frac{1}{4}\right)^n u[n]$
- $|z| < \frac{1}{4} \rightarrow h[n] = -\left(\frac{1}{2}\right)^n u[-n-1] - 2\left(-\frac{1}{4}\right)^n u[-n-1]$
- $\frac{1}{4} < |z| < \frac{1}{2} \rightarrow h[n] = -\left(\frac{1}{2}\right)^n u[-n-1] + 2\left(-\frac{1}{4}\right)^n u[n]$



CONSIDER

$$h[n] = \left(\frac{1}{2}\right)^n u[n] + 2\left(-\frac{1}{4}\right)^n u[n]$$

$$\frac{1}{8} + \frac{-1}{\cancel{64}\,32} = 3/32$$

n	$h[n]$
0	3
1	0
2	3/8
3	3/32

CASE III $M \geq N$, + POLES IN UNIQUE LOCATIONS

CONSIDER $H(z) = \dfrac{5 + z^{-1} + 4z^{-2} + 3z^{-3}}{1 - 3z^{-1}}$

ASSUME $H(z)$ STABLE

$M = 3$

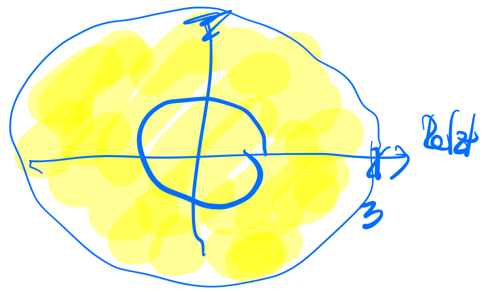
$N = 1$

USE LONG DIVISION

$$
\begin{array}{r|l}
 & -z^{-2} - \frac{5}{3}z^{-1} - \frac{8}{9} \\
\hline
-3z^{-1}+1 & 3z^{-3} + 4z^{-2} + z^{-1} + 5 + 0z \\
 & 3z^{-3} - z^{-2} \\
\hline
 & 5z^{-2} + z^{-1} \\
 & 5z^{-2} - \frac{5}{3}z^{-1} \\
\hline
 & \frac{8}{3}z^{-1} + 5 \\
 & \frac{8}{3}z^{-1} - \frac{8}{9} \\
\hline
 & \frac{53}{9}
\end{array}
$$

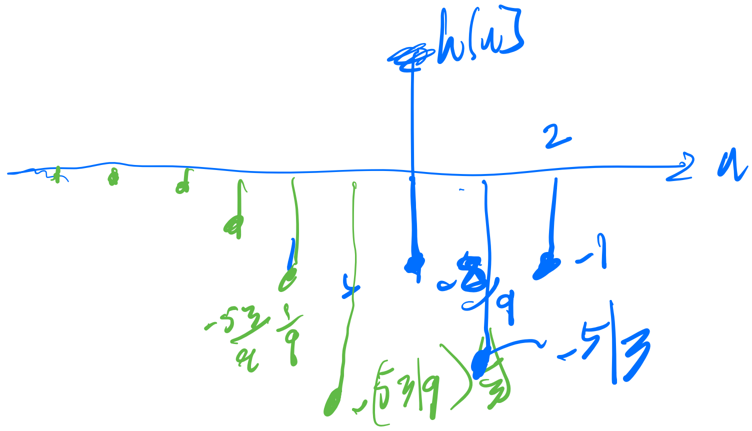
ROC

$|z| < 3$



$$H(z) = \underbrace{-z^{-2} - \frac{5}{3}z^{-1} - \frac{8}{9}} + \frac{53/9}{1 - 3z^{-1}}, \quad |z| < 3$$

$$h[n] = -\frac{8}{9}\delta[n] - \frac{5}{3}\delta[n-1] - \delta[n-2] - \frac{53}{9}(3)^n u[-n-1]$$



CASE III POLES IN SAME LOCATION

CONSIDER S POLES @ $z = d_i$, OTHER POLES UNIQUE

IN GENERAL

$$H(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{\substack{k=1 \\ k \neq i}}^{N} \frac{A_k}{1 - d_k z^{-1}} + \sum_{m=1}^{S} \frac{C_m}{(1 - d_i z^{-1})^m}$$

(CASE II) VIA LONG DIVISION

(CASE I)

CASE III

$$C_m = \frac{1}{(S-m)!\,(-d_i)^{S-m}} \left\{ \frac{d^{S-m}}{dw^{S-m}} (1 - d_i w)^S H(w^{-1}) \right\} \Bigg|_{w = d_i^{-1}}$$

EX: $H(z) = \frac{1}{(1 - \frac{1}{2} z^{-1})^2 (1 + \frac{1}{4} z^{-1})} = \frac{A_1}{1 + \frac{1}{4} z^{-1}} + \frac{C_1}{1 - d_i z^{-1}} + \frac{C_2}{(1 - d_i z^{-1})^2}$

$$A_1 = H(z)(1 - d_k z^{-1}) \Big|_{z = d_k} = \frac{1}{(1 - \frac{1}{2} z^{-1})^2} \Bigg|_{z = -\frac{1}{4}} = \frac{1}{(1 - \frac{-4}{2})^2} = \frac{1}{9}$$

$$C_m = \frac{1}{(S-m)!\,(-d_i)^{S-m}} \left\{ \frac{d^{S-m}}{dw^{S-m}} (1 - d_i w)^S H(w^{-1}) \right\} \Bigg|_{w = d_i^{-1}}$$

$s=2,\ d_i=\frac{1}{2}$

For $m=1$

$$C_1=\frac{1}{1!\left(-\frac{1}{2}\right)^1}\left\{\frac{d}{dw}\cancel{\left(1-\frac{1}{2}w\right)^2}\frac{1}{\cancel{\left(1-\frac{1}{2}w\right)^2}}\frac{1}{\left(1+\frac{1}{4}w\right)}\right\}_{w=2}$$

$$=(-2)\left[\frac{-\left(\frac{1}{4}\right)}{\left(1+\frac{1}{4}w\right)^2}\right]\Bigg|_{w=2}=\frac{(-2)\left(-\frac{1}{4}\right)}{\left(\frac{3}{2}\right)^2}=\frac{1}{2}\cdot\frac{4}{9}$$

$$=\frac{2}{9}$$

$$C_m=\frac{1}{(s-m)!\,(-d_i)^{s-m}}\left\{\frac{d^{s-m}}{dw^{s-m}}(1-d_iw)^s H(w^{-1})\right\}\Bigg|_{w=d_i^{-1}}$$

$m=2,\ s=2,\ d_i=\frac{1}{2}$

$$C_2=\frac{1}{1\cancel{(-d_i)^0}}=\cancel{\left(1-\frac{1}{2}w\right)^2}\frac{1}{\cancel{\left(1-\frac{1}{2}w\right)^2}}\frac{1}{\left(1+\frac{1}{4}w\right)}\Bigg|_{w=2}=\frac{2}{3}$$

$$H(z)=\frac{\frac{1}{9}}{1+\frac{1}{4}z^{-1}}+\frac{\frac{2}{9}}{1-\frac{1}{2}z^{-1}}+\frac{\frac{2}{3}}{\left(1-\frac{1}{2}z^{-1}\right)^2},\qquad |z|>\frac{1}{2}$$

CORRECTED + COMPLETED SOLUTION (POST LECTURE)

WHAT IS THE IZT OF $\frac{2/3}{\left(1-\frac{1}{2}z^{-1}\right)^2}$, $|z| > \frac{1}{2}$???

AS USUAL, WE KNOW $\alpha^n u[n] \Leftrightarrow \frac{1}{1-\alpha z^{-1}}$, $|z| > |\alpha|$

IN GENERAL, ZT IS $X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$

TAKE $\frac{d}{dz}$ OF BOTH SIDES:

$$\frac{dX(z)}{dz} = \sum_{n=-\infty}^{\infty} x[n] \frac{d}{dz}\left(z^{-n}\right)$$

$$= \sum_{n=-\infty}^{\infty} x[n] (-n) z^{-n-1}$$

$$\frac{dX(z)}{dz} = -z^{-1} \sum_{n=-\infty}^{\infty} n\, x[n] z^{-n}$$

So.... $n\,x[n] \Leftrightarrow -z \frac{dX(z)}{dz}$

LET'S APPLY THIS PROPERTY TO

$$\alpha^n u[n] \Leftrightarrow \frac{1}{1-\alpha z^{-1}}$$

$$n\,\alpha^n u[n] \Leftrightarrow -z \underbrace{\left(\frac{\cancel{\left(1-\alpha z^{-1}\right)(0)} - (1)\left(\alpha z^{-2}\right)}{\left(1-\alpha z^{-1}\right)^2}\right)}_{\frac{dX(z)}{dz}}$$

OR $n\alpha^n u[n] \Leftrightarrow \dfrac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$

APPLYING SHIFT + MULTIPLY PROPERTIES $= \frac{1}{\alpha}\cdot(n+1)\,\alpha^{n+1}u[n+1] \Leftrightarrow \dfrac{1}{(1-\alpha z^{-1})^2}$

$= (n+1)\,\alpha^n u[n+1]$

$= (n+1)\,\alpha^n u[n] \Leftrightarrow \dfrac{1}{(1-\alpha z^{-1})^2}$

BECAUSE $(n+1) = 0$ FOR $n = -1$

SO... THE COMPLETE SOLUTION IS

$$H(z) = \frac{\frac{1}{9}}{1+\frac{1}{4}z^{-1}} + \frac{\frac{2}{9}}{1-\frac{1}{2}z^{-1}} + \frac{\frac{2}{3}}{(1-\frac{1}{2}z^{-1})^2}, \quad |z| > \frac{1}{2}$$

$$h[n] = \frac{1}{9}\left(-\frac{1}{4}\right)^n u[n] + \frac{2}{9}\left(\frac{1}{2}\right)^n u[n] + \frac{2}{3}(n+1)\left(\frac{1}{2}\right)^n u[n]$$

$$= \frac{1}{9}\left(-\frac{1}{4}\right)^n u[n] + \frac{8}{9}\left(\frac{1}{2}\right)^n u[n] + \frac{2}{3}\,n\left(\frac{1}{2}\right)^n u[n]$$