4.5 THE INVERSE z-TRANSFORM USING CONTOUR INTEGRATION

In Section 4.3 we discussed a number of procedures for obtaining the sequence associated with a given z-transform expression. In this section we develop a formal expression for the inverse z-transform. This expression is used in Sections 4.6 and 4.7 to develop two additional properties of the z-transform, specifically the complex convolution theorem and Parseval's relation for the z-transform.

A formal inverse z-transform relation can be derived using the Cauchy integral theorem (see Churchill and Brown, 1984). This theorem states that

$$\frac{1}{2\pi j}\oint_C z^{-k}\,dz = \begin{cases} 1, & k = 1, \\ 0, & k \neq 1, \end{cases} \tag{4.63}$$

where C is a counterclockwise contour that encircles the origin.

The z-transform relation is

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}. \tag{4.64}$$

Multiplying both sides of Eq. (4.64) by z^{k-1} and integrating with a contour integral for which the contour of integration encloses the origin and lies entirely in the region of convergence of $X(z)$, we obtain

$$\frac{1}{2\pi j}\oint_C X(z)z^{k-1}\,dz = \frac{1}{2\pi j}\oint_C \sum_{n=-\infty}^{\infty} x[n]z^{-n+k-1}\,dz. \tag{4.65}$$

Interchanging the order of integration and summation on the right-hand side of Eq. (4.65) (valid if the series is convergent), we obtain

$$\frac{1}{2\pi j}\oint_C X(z)z^{k-1}\,dz = \sum_{n=-\infty}^{\infty} x[n]\frac{1}{2\pi j}\oint_C z^{-n+k-1}\,dz, \tag{4.66}$$

which from Eq. (4.63) becomes

$$\frac{1}{2\pi j}\oint_C X(z)z^{k-1}\,dz = x[k].$$

Therefore, the inverse z-transform relation is given by the contour integral

$$x[n] = \frac{1}{2\pi j}\oint_C X(z)z^{n-1}\,dz, \tag{4.67}$$

where C is a counterclockwise closed contour in the region of convergence of $X(z)$ and encircling the origin of the z-plane. It should be stressed that in deriving Eq. (4.67) we made no assumption about whether k in Eq. (4.65) or n in Eq. (4.67) was positive or negative, and consequently Eq. (4.67) is valid for both positive and negative values of n.

Equation (4.67) is the formal inverse z-transform expression. If the region of convergence includes the unit circle and if the contour of integration is taken to be the unit circle, then on this contour, $X(z)$ reduces to the Fourier transform and Eq. (4.67) reduces to the inverse Fourier transform expression, Eq. (2.112). Specifically, with $z = e^{j\omega}$, Eq. (4.67) becomes

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}\,d\omega, \tag{4.68}$$

where we have used the fact that integrating in z counterclockwise around the unit circle is equivalent to integrating in ω from $-\pi$ to $+\pi$ and that $dz = je^{j\omega}\,d\omega$.

Contour integrals of the form of Eq. (4.67) are often conveniently evaluated using Cauchy's residue theorem which, when applied to Eq. (4.67), gives

$$\begin{aligned} x[n] &= \frac{1}{2\pi j}\oint_C X(z)z^{n-1}\,dz \\ &= \sum[\text{residues of } X(z)z^{n-1} \text{ at the poles inside } C]. \end{aligned} \tag{4.69}$$

Equation (4.69) is valid for any proper z-transform $X(z)$, but finding residues of nonrational functions is often difficult. However, if $X(z)z^{n-1}$ is a rational function of z, it may be expressed as

$$X(z)z^{n-1} = \frac{\psi(z)}{(z-d_0)^s}, \tag{4.70}$$

where $X(z)z^{n-1}$ has s poles at $z = d_0$ and $\psi(z)$ has no poles at $z = d_0$. The residue of $X(z)z^{n-1}$ at $z = d_0$ is given by

$$\text{Res}[X(z)z^{n-1} \text{ at } z = d_0] = \frac{1}{(s-1)!}\left[\frac{d^{s-1}\psi(z)}{dz^{s-1}}\right]_{z=d_0}. \tag{4.71}$$

In particular, if there is only a first-order pole at $z = d_0$, i.e., if $s = 1$, then

$$\text{Res}[X(z)z^{n-1} \text{ at } z = d_0] = \psi(d_0). \tag{4.72}$$

It is clear from a comparison of Eq. (4.44) and Eq. (4.71) that finding residues of $X(z)z^{n-1}$ is much like finding coefficients for a partial fraction expansion of $X(z)$. Generally, for rational functions it will be easier to use the method developed in Section 4.3.2; however, we will illustrate the use of Eq. (4.69) with the simple rational function of Example 4.19. We will also find the general expression of Eq. (4.67) to be useful in later chapters.

Example 4.19

As an example of the use of contour integration to evaluate the inverse transform relation, let us consider the inverse transform of

$$X(z) = \frac{1}{1-az^{-1}}, \qquad |z| > |a|.$$

Using Eq. (4.67), we obtain

$$x[n] = \frac{1}{2\pi j}\oint_C \frac{z^{n-1}}{1-az^{-1}}\,dz = \frac{1}{2\pi j}\oint_C \frac{z^n\,dz}{z-a},$$

where the contour of integration, C, is a circle of radius greater than a. For $n \geq 0$, the contour of integration encloses only one pole at $z = a$. Consequently, for $n \geq 0$, $x[n]$ is given by

$$x[n] = a^n, \qquad n \geq 0.$$

For $n < 0$, there is a multiple-order pole at $z = 0$ whose order depends on n. For $n = -1$, the pole is first order with a residue of $-a^{-1}$. The residue of the pole at $z = a$ is a^{-1}. Consequently, the sum of the residues is zero, and hence $x[-1] = 0$. For $n = -2$,

$$\text{Res}\left[\frac{1}{z^2(z-a)} \text{ at } z = a\right] = a^{-2}$$

and

$$\text{Res}\left[\frac{1}{z^2(z-a)} \text{ at } z = 0\right] = -a^{-2},$$

and thus $x[-2] = 0$. By continuing this procedure it can be verified that for this example $x[n] = 0$, $n < 0$. As n becomes more negative, the evaluation of the residue of the multiple-order pole at $z = 0$ becomes increasingly tedious.

While Eq. (4.67) is valid for all n, its use for $n < 0$ is often cumbersome because of the multiple-order poles at $z = 0$.

This can be avoided by modifying Eq. (4.67) by means of a substitution of variables, making the residue theorem easier to apply for $n < 0$. Specifically, consider the substitution of variables $z = p^{-1}$, so that Eq. (4.67) becomes

$$x[n] = \frac{-1}{2\pi j}\oint_{C'} X(1/p)p^{-n+1}p^{-2}\,dp. \tag{4.73}$$

Observe that since the contour in Eq. (4.67) is a counterclockwise contour, the contour in Eq. (4.73) is a clockwise contour. Multiplying by -1 to reverse the direction of the contour, the above substitution of variables then leads to the expression

$$\begin{aligned} x[n] &= \frac{1}{2\pi j}\oint_{C'} X(1/p)p^{-n-1}\,dp \\ &= \sum [\text{residues of } X(1/p)p^{-n-1} \text{ at the poles inside } C']. \end{aligned} \tag{4.74}$$

If the contour C in Eq. (4.67) is a circle of radius r in the z-plane, then the contour C' in Eq. (4.74) is a circle of radius $1/r$ in the p-plane. The poles of $X(z)$ that were outside the contour C correspond now to poles of $X(1/p)$ that are inside the contour C', and vice versa. Additional poles or zeros may or may not appear at the origin or at infinity, but that is not crucial to the argument. In example 4.19, $x[n]$ can also be expressed as

$$x[n] = \frac{1}{2\pi j}\oint_{C'} \frac{p^{-n-1}}{1 - ap}\,dp. \tag{4.75}$$

The contour of integration C' is now a circle of radius less than $1/a$. For $n < 0$ there are no singularities inside the contour of integration, so $x[n] = 0$ for $n < 0$. Just as Eq. (4.69) was cumbersome (although certainly valid) for evaluating $x[n]$ for $n < 0$, the expression in Eq. (4.74) is likewise cumbersome (but still valid) for evaluating $x[n]$ for $n \geq 0$ because of the multiple-order poles that appear at the origin.