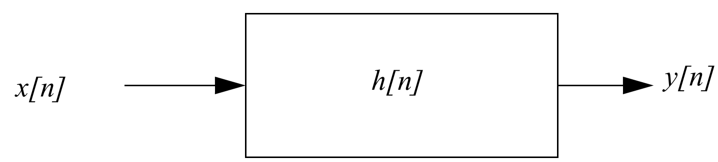
Fundamentals of Signal Processing (18-491)
Spring Semester, 2019

SOLUTION OF DIFFERENCE EQUATIONS

Difference equations are a complementary way of characterizing the response of LSI systems (along with their impulse responses and various transform-based characterizations. In these notes we review how they are solved in discrete time using a simple example.

Note: analytical solution to difference equations will not appear on the homework or the exam. These notes are provided for your information only.



Introduction. Difference equations in discrete-time systems play the same role in characterizing the time-domain response of discrete-time LSI systems that differential equations play for continuous-time LTI systems. In the most general form we can write difference equations as

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]$$

where (as usual) $x[n]$ represents the input and $y[n]$ represents the output. Since we can set a_0 equal to 0 without any loss of generality, we can rewrite this as

$$y[n] = -\sum_{k=1}^{N} a_k y[n-k] + \sum_{m=0}^{M} b_m x[n-m]$$

In this representation we characterize the present output of an LSI system as a linear combination of past outputs combined with a linear combination of the present and previous inputs. As before, the difference equations alone do not uniquely specify the system. Initial conditions are needed as well. Normally we assume initial rest (*i.e.* the output is zero before the input is applied). Otherwise the system would be neither linear nor shift-invariant. (Why?)

Difference equations can be solved analytically, just as in the case of ordinary differential equations. As before, the solution involves obtaining the homogenous solution (or the natural frequencies) of the system, and the particular solution (or the forced response).

In this handout we consider the specific example of the simple difference equation:

$$y[n] - \frac{1}{4}y[n-1] - \frac{1}{8}y[n-2] = 3x[n] - \frac{3}{4}x[n-1]$$

or,

$$y[n] = \frac{1}{4}y[n-1] + \frac{1}{8}y[n-2] + 3x[n] - \frac{3}{4}x[n-1]$$

Solution by iteration. Let's first obtain the solution of this equation via iteration. This can be done by setting up a table of variables, entering the known entries for $x[n]$ and $x[n-1]$ for all n, plus the initial conditions $y[n-1] = y[n-2] = 0$ for $n = 0$ and $y[n-2] = 0$ for $n = 1$, and then filling in the remaining entries row by row:

n	$x[n]$	$x[n-1]$	$y[n-2]$	$y[n-1]$	$y[n]$
0	1	0	0	0	3
1	1/3	1	0	3	1
2	1/9	1/3	3	1	17/24
3	1/27	1/9	1	17/24	etc.
4	1/81	1/27	17/24		

We see, for example, that $y[0] = 3$ and $y[1] = 1$.

The homogeneous solution. As before, the homogeneous solution $y_h[n]$ is obtained by assuming solutions of an exponential form, although this time we will use $y_h[n] = A\alpha^n$. Again, the original equation is solved with the input $x(t)$ equal to zero:

$$y_h[n] - \frac{1}{4}y_h[n-1] - \frac{1}{8}y_h[n-2] = 0$$

$$A\alpha^n - \frac{1}{4}A\alpha^{n-1} - \frac{1}{8}A\alpha^{n-2} = 0$$

Dividing by the common term $A\alpha^n$ and solving for α we obtain the *characteristic equation*

$$1 - \frac{1}{4}\alpha^{-1} - \frac{1}{8}\alpha^{-2} = 0 \text{ or}$$

$$\alpha^2 - \frac{1}{4}\alpha - \frac{1}{8} = 0$$

which solves to

$$\left(\alpha + \frac{1}{4}\right)\left(\alpha - \frac{1}{2}\right) = 0 \text{ or } \alpha = -\frac{1}{4} \text{ and } \alpha = \frac{1}{2}$$

The most general form of the solution would be a linear combination of the two terms:

$$y_h[n] = A_1\left(-\frac{1}{4}\right)^n + A_2\left(\frac{1}{2}\right)^n \text{ for } n > 0$$

The particular solution. As before, particular solution is obtained by assuming an output that is proportional to the input. Here we would have

$$y_p[n] = K\beta^n$$

Unfortunately, the initial conditions can not be applied as easily for the case of discrete-time systems as they are for continuous-tome systems.In the case of the differential equations, the initial conditions are that the output and all its derivatives are zero when the input is applied. In the case of the difference equations the constraints are that all values of the output are zero when the input is applied. (Actually it is only necessary for the previous N values to be zero. In the example given, $N = 2$ so the initial conditions are $y[n-1] = y[n-2] = 0$. The problem arises because the total system input is (*i.e.* the right side of the difference equation) is

$$3\left(\frac{1}{3}\right)^n u[n] - \frac{3}{4}\left(\frac{1}{3}\right)^{n-1} u[n-1]$$

This is an issue because while the first term is valid for $n \geq 0$, the second term is valid only for $n \geq 1$. Hence a single set of initial conditions cannot be applied at $n = 0$.

Following the method discussed in the lecture, we first solve for a simple exponential input and then obtain the solution we really want via superposition. Specifically, we solve the equation

$$v[n] - \frac{1}{4}v[n-1] - \frac{1}{8}v[n-2] = \left(\frac{1}{3}\right)^n u[n]$$

The solution to our original equation is then obtained via superposition:

$$y[n] = 3v[n] - \frac{3}{4}v[n-1]$$

Note that $v[n] = v_h[n] + v_p[n]$ and $v_h[n] = y_h[n]$ which we have already obtained.

The particular solution can be easiliy obtained by letting

$$v_p[n] = K\left(\frac{1}{3}\right)^n$$

Plugging in to the original equation we obtain

$$v_p[n] - \frac{1}{4}v_p[n-1] - \frac{1}{8}v_p[n-2] = \left(\frac{1}{3}\right)^n u[n] \text{ or}$$

$$K\left(\frac{1}{3}\right)^n - \frac{1}{4}K\left(\frac{1}{3}\right)^{n-1} - \frac{1}{8}K\left(\frac{1}{3}\right)^{n-2} = \left(\frac{1}{3}\right)^n$$

Dividing through by the common factor $\left(\frac{1}{3}\right)^n$ we obtain

$$K - \frac{1}{4}(3)K - \frac{1}{8}(9)K = 1 \text{ or}$$

$$K = \frac{1}{1 - \frac{3}{4} - \frac{9}{8}} = -\frac{8}{7} \text{ and}$$

$$v_p[n] = -\frac{8}{7}\left(\frac{1}{3}\right)^n$$

Hence, the total solution is

$$v[n] = A_1\left(-\frac{1}{4}\right)^n + A_2\left(\frac{1}{2}\right)^n - \frac{8}{7}\left(\frac{1}{3}\right)^n$$

Now we can apply the initial conditions $v[-1] = v[-2] = 0$.

Solving for $n = -1$ and $n - 2$ in turn, we obtain

$$A_1\left(-\frac{1}{4}\right)^{-1} + A_2\left(\frac{1}{2}\right)^{-1} - \frac{8}{7}\left(\frac{1}{3}\right)^{-1} = 0 \text{ and}$$

$$A_1\left(-\frac{1}{4}\right)^{-2} + A_2\left(\frac{1}{2}\right)^{-2} - \frac{8}{7}\left(\frac{1}{3}\right)^{-2} = 0$$

or ...

$$-4A_1 + 2A_2 = \frac{24}{7} \text{ and}$$

$$16A_1 + 4A_2 = \frac{72}{7}$$

Solving this system of equations yields $A_1 = \frac{1}{7}$ and $A_2 = 2$.

So the complete solution for $v[n]$ is

$$v[n] = \left[\frac{1}{7}\left(-\frac{1}{4}\right)^n + 2\left(\frac{1}{2}\right)^n - \frac{8}{7}\left(\frac{1}{3}\right)^n\right]u[n]$$

And now we can (finally) obtain our original solution as

$$y[n] = 3v[n] - \frac{3}{4}v[n-1] \text{ or}$$

$$y[n] = 3\left\{\left[\frac{1}{7}\left(-\frac{1}{4}\right)^n + 2\left(\frac{1}{2}\right)^n - \frac{8}{7}\left(\frac{1}{3}\right)^n\right]u[n]\right\} - \frac{3}{4}\left\{\left[\frac{1}{7}\left(-\frac{1}{4}\right)^{n-1} + 2\left(\frac{1}{2}\right)^{n-1} - \frac{8}{7}\left(\frac{1}{3}\right)^{n-1}\right]u[n-1]\right\}$$

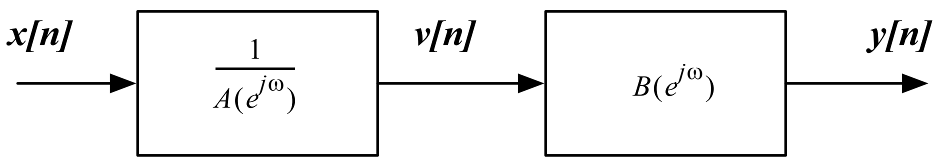
Note that $y[0] = 3$ and $y[1] = 1$ as obtained originally by iteration.

This is a lot of work! We can obtain the same result much more easily later via transform techniques: just take the z-transform of $x[n]$ and $h[n]$, multiply them together, and compute the inverse z-transform.

Frequency domain justification for the use of superposition. It is easy to show that the transfer function corresponding to the system that is specified by the difference equation for the example above is

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{3 - \frac{3}{4}e^{-j\omega}}{1 - \frac{1}{4}e^{-j\omega} - \frac{1}{8}e^{-j2\omega}} \equiv \frac{B(e^{j\omega})}{A(e^{j\omega})}$$

Now suppose that we separated the numerator and denominator components of the transfer function as follows:



In other words, $V(e^{j\omega}) = \dfrac{X(e^{j\omega})}{A(e^{j\omega})}$ and $Y(e^{j\omega}) = B(e^{j\omega})V(e^{j\omega})$. It can be easily seen that $Y(e^{j\omega})$ is still equal to $H(e^{j\omega})X(e^{j\omega})$ as before. Cross-multiplying and taking the inverse transform of the equations for $V(e^{j\omega})$ and $Y(e^{j\omega})$ at the beginning of the paragraph produces almost by inspection the difference equations

$$v[n] - \frac{1}{4}v[n-1] - \frac{1}{8}v[n-2] = x[n] \text{ and}$$

$$y[n] = 3v[n] - \frac{3}{4}v[n-1]$$