ZT PROPERTIES,

ZTs + LSI SYSTEMS,

INVERSE ZTs

DTFT

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega$$

FOR $\sum_n |x[n]|^2 < \infty$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

ZT

$$x[n] = \frac{1}{2\pi j}\oint X(z)\, z^{n-1}\, dz$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

+ ROC

Properties of Z-Transforms

Linearity $\quad a x_1[n] + b x_2[n] \Longleftrightarrow a X_1(z) + b X_2(z)$

ROC $R_{x_1} \cap R_{x_2}$

Time shift $\quad x[n-N] \Longleftrightarrow z^{-N} X(z)$, R_x (except maybe not $z=0$)

Multiplication by complex exponential $\quad z_0^n x[n] \Longleftrightarrow X\left(\frac{z}{|z_0|}\right)$, ROC $|z_0| \cdot R_x$

Convolution in time $\quad x[n] * h[n] \Longleftrightarrow H(z) \cdot X(z)$ ROC $R_x \cap R_h$

Mult in time $\quad x[n] \cdot w[n] \Longleftrightarrow \frac{1}{2\pi j} \oint X(v)\, W\left(\frac{z}{v}\right) v^{-1}\, dv$

ZTs + LSI Systems

$$x[n] \to \boxed{h[n]} \to y[n]$$

$$Y(z) = X(z) \cdot H(z)$$

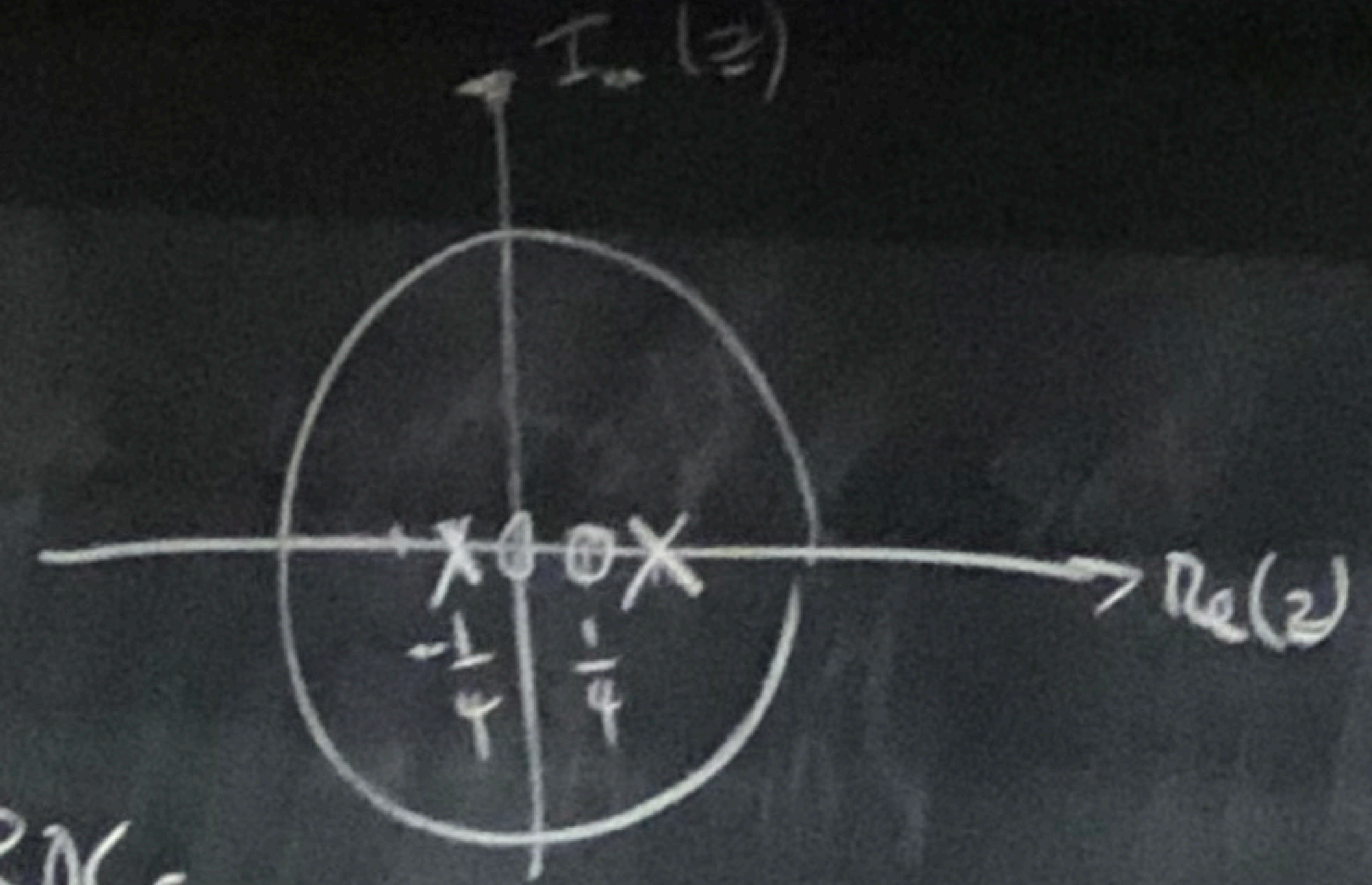
Consider

$$H(z) = \frac{3 - \frac{3}{4} z^{-1}}{1 - \frac{1}{4} z^{-1} - \frac{1}{8} z^{-2}} = \frac{3 - \frac{3}{4} z^{-1}}{\left(1 - \frac{1}{2} z^{-1}\right)\left(1 + \frac{1}{4} z^{-1}\right)}$$

$$= \frac{3z\left(z - \frac{1}{4}\right)}{\left(z - \frac{1}{2}\right)\left(z + \frac{1}{4}\right)}$$

DIFF. of ZT

$n\,x[n] \Leftrightarrow -z \dfrac{dX(z)}{dz}$ USUALLY R_x



POSSIBLE ROCs

$|z| < \frac{1}{4} \Rightarrow$ LEFT-SIDED $h[n]$

$\frac{1}{4} < |z| < \frac{1}{2} \Rightarrow$ BOTH-SIDED $h[n]$

$\frac{1}{2} < |z| \Rightarrow$ RIGHT-SIDED $h[n]$

$$H(z) = \frac{3 - \frac{3}{4} z^{-1}}{1 - \frac{1}{4} z^{-1} - \frac{1}{8} z^{-2}} = \frac{Y(z)}{X(z)}$$

$$Y(z)\left(1 - \tfrac{1}{4} z^{-1} - \tfrac{1}{8} z^{-2}\right) = X(z)\left(3 - \tfrac{3}{4} z^{-1}\right)$$

DIFFERENCE EQ.

$$y[n] - \tfrac{1}{4} y[n-1] - \tfrac{1}{8} y[n-2] = 3x[n] - \tfrac{3}{4} x[n-1]$$

$$y[n] = \tfrac{1}{4} y[n-1] + \tfrac{1}{8} y[n-2] + 3x[n] - \tfrac{3}{4} x[n-1]$$

IN GENERAL

$$Y(z)\sum_{k=0}^{N} a_k z^{-k} = X(z)\sum_{\ell=0}^{M} b_\ell z^{-\ell}$$

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{\ell=0}^{M} b_\ell x[n-\ell]$$

$$a_0 = 1 \Rightarrow y[n] = -\sum_{k=1}^{N} a_k y[n-k] + \sum_{\ell=0}^{M} b_\ell x[n-\ell]$$

HOW CAN WE OBTAIN h[n] FROM H(z)?

EASIEST SOLUTION: LET $x[n] = \delta[n]$, FIND SOLUTION BY ITERATION

FIND $y[n]$ GIVEN $x[n] = \delta[n]$ ...

IC's: $y[-1] = y[-2] = 0$

	$-3/4$	3	$+1/8$	$+1/4$	
n	$x[n-1]$	$x[n]$	$y[n-2]$	$y[n-1]$	$y[n]$
0	0	1	0	0	3
1	1	0	0	3	0
2	0	0	3	0	$3/8$
3	0	0	0	$3/8$	$3/32$

EVALUATING INVERSE Z-TRANSFORMS

POSSIBLE METHODS:

X 1. DIRECT EVALUATION OF COMPLEX CONTOUR INTEGRAL
2. LONG DIVISION
★ 3. PARTIAL FRACTION EXPANSION
4. TAYLOR SERIES EXPANSION

$$3 + 0z^{-1} + \frac{3}{8}z^{-2} + \frac{3}{32}z^{-3} + \ldots$$

$$-\frac{3}{4}z^{-1} + 0z^{-2} + 0z^{-3}$$

$$\frac{3}{4}z^{-1} - \frac{3}{8}z^{-2}$$

$$0z^{-1} + \frac{3}{8}z^{-2}$$

$$0z^{-1} + 0z^{-2} + 0z^{-3}$$

$$\frac{3}{8}z^{-2} + 0z^{-3} + 0z^{-4}$$

$$\frac{3}{8}z^{-2} - \frac{3}{32}z^{-3} - \frac{3}{64}z^{-4}$$

$$\frac{3}{32}z^{-3} + \frac{3}{64}z^{-4} + 0z^{-5}$$

$$h[n] = 3\delta[n] + 0\delta[n-1] + \frac{3}{8}\delta[n-2] + \frac{3}{32}\delta[n-3] + \ldots$$

ZT PROPERTIES,

ZTs + LSI SYSTEMS,

INVERSE ZTs

(OS 4P 3.3 - 3.5)

$$6z + 36z$$

$$-\frac{1}{8}z^{-2} - \frac{1}{4}z^{-1} + 1 \,\big)\, -\frac{3}{4}z^{-1} + 3 + 0z + 0z^{2}$$

$$-\frac{3}{4}z^{-1} - \frac{3}{2}z + 0z$$

$$\frac{9}{2}z^{-1} + 0z$$

PARTIAL FRACTION EXPANSION

$$H(z) = \frac{3 - \frac{3}{4}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{4}z^{-1}\right)} = \frac{A_1}{1-\frac{1}{2}z^{-1}} + \frac{A_2}{1+\frac{1}{4}z^{-1}}$$

CASES:
1. $M < N$, UNIQUE POLE LOCATIONS
2. $M \geq N$, UNIQUE POLE LOCATIONS
3. MULTIPLES IN ONE (OR MORE) LOCATION

$$H(z) = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{2}{1+\frac{1}{4}z^{-1}} = \frac{3 - \frac{3}{4}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{4}z^{-1}\right)}$$

CASE 1:

$$H(z) = \sum_{i=1}^{N} \frac{A_i}{1-d_i z^{-1}}$$

A_i RESIDUES

d_i POLE LOCATIONS

$$A_i = H(z)\left(1-d_i z^{-1}\right)\Big|_{z=d_i}$$

$$A_1 = H(z)\left(1-\frac{1}{2}z^{-1}\right)\Big|_{z=\frac{1}{2}} = \frac{3-\frac{3}{4}z^{-1}}{1+\frac{1}{4}z^{-1}}\Big|_{z=} = \frac{3-\frac{3}{4}\cdot 2}{1+\frac{1}{4}\cdot 2} = \frac{3/2}{3/2} = 1$$

$$A_2 = \frac{3-\frac{3}{4}z^{-1}}{1-\frac{1}{2}z^{-1}}\Big|_{z=-\frac{1}{4}} = \frac{3+\frac{3\cdot 4}{4}}{1+\frac{4}{2}} = \frac{6}{3} = 2$$

$$\alpha^n u[n] \Leftrightarrow \frac{1}{1-\alpha z^{-1}}, \ |z|>|\alpha|$$

$$-\alpha^n u[-n-1] \Leftrightarrow \frac{1}{1-\alpha z^{-1}}, \ |z|<|\alpha|$$

DTFT

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n}\, d\omega$$

FOR $\sum_n |x[n]|^2 < \infty$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

ZT

$$x[n] = \frac{1}{2\pi j}\oint X(z) z^{n-1}\, dz$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

+ ROC

EVALUATING INVERSE Z-TRANSFORMS

ATION

$$H(z) = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{2}{1+\frac{1}{4}z^{-1}}$$

CASE 3:

$h[0] = 3$

$h[1] = 0$

$h[2] = 3/8$

SOLUTIONS:

1. $|z| < \frac{1}{4}$: $h[n] = -\left(\frac{1}{2}\right)^n u[-n-1] - 2\left(-\frac{1}{4}\right)^n u[-n-1]$

2. $\frac{1}{4} < |z| < \frac{1}{2}$: $h[n] = -\left(\frac{1}{2}\right)^n u[-n-1] + 2\left(-\frac{1}{4}\right)^n u[n]$

3. $|z| > \frac{1}{2}$: $h[n] = \left(\frac{1}{2}\right)^n u[n] + 2\left(-\frac{1}{4}\right)^n u[n]$