1/31/24

Z-TRANSFORM PROPERTIES, LSI SYSTEMS, DIFFERENCE EQS., + INVERSES

(OS4E 3.0-3.5, NOTES)

DTFT $x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$

$\{e^{j\omega n}\}$

$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$

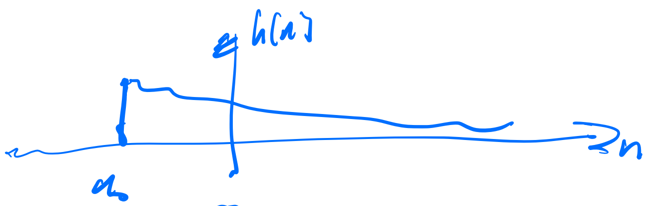
$\{z^n = \rho^n e^{j\omega n}\}$ ZT $x[n] = \frac{1}{2\pi j}\oint X(z) z^{n-1} dz$

$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$

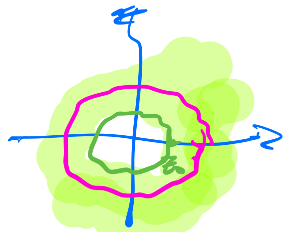
$x[n], h[n]$ + ROC....

IF LSI SYSTEM IS CAUSAL $h[n]=0,\ n<0$

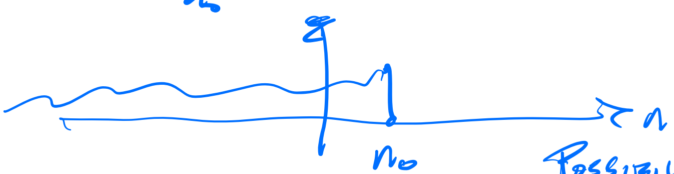
RIGHT-SIDED $h[n]$



ROC $|z|>|z_0|$

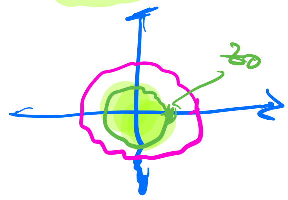


LEFT-SIDED

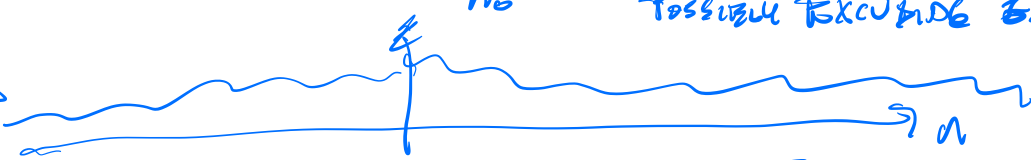


$|z|<|z_0|$

POSSIBLY EXCLUDING $z=0$

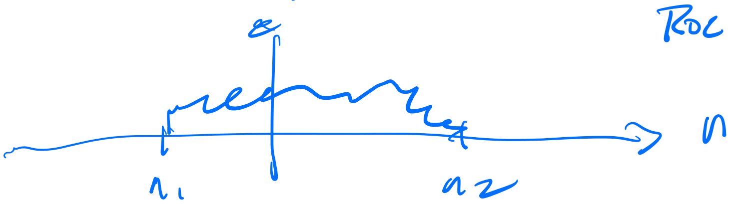


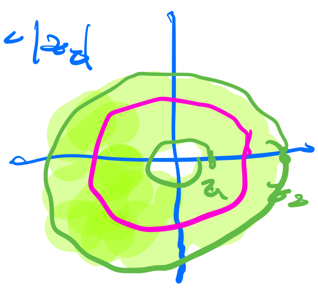
BOTH-SIDED



ROC $|z_1|<|z|<|z_2|$

FINITE-DURATION





STABILITY + THE ROC

FOR LSI SYSTEM TO BE STABLE $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$

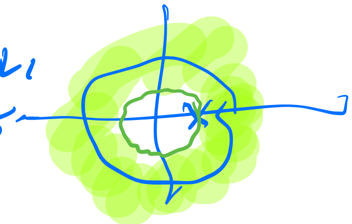
FOR DTFT TO EXIST ROC MUST INCLUDE $|z|=1$ $\sum_{n=-\infty}^{\infty} |h[n]|^2 < \infty$

IF LSI SYSTEM IS STABLE, ROC MUST INCLUDE UNIT CIRCLE

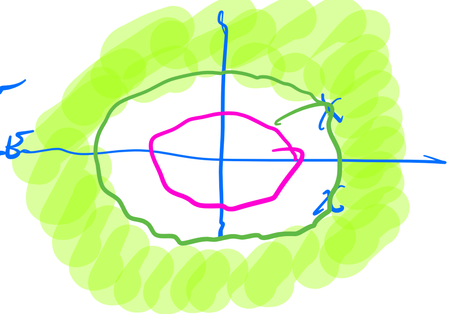
FOR LSI SYSTEM TO BE CAUSAL AND STABLE,

ALL POLES MUST BE

CAUSAL, STABLE



CAUSAL UNSTABLE



PROPERTIES of THE ZT

$$x[n] \Longleftrightarrow X(z), \quad R_x$$

LINEARITY $\quad a\,x_1[n] + b\,x_2[n] \Longleftrightarrow a\,X_1(z) + b\,X_2(z), \quad R_{x_1} \cap R_{x_2}$

TIME SHIFT $\quad x[n-N] \Longleftrightarrow X(z)\,z^{-N}, \quad R_x$

CONVOLUTION IN TIME $\quad x[n] * h[n] \Longleftrightarrow X(z)\cdot H(z) \quad R_x \cap R_h$

MULT IN TIME $\quad x_1[n]\cdot x_2[n] \Longleftrightarrow \frac{1}{2\pi j}\oint_C X_1(\nu)\,X_2\!\left(\frac{z}{\nu}\right) d\nu$

$$R_{x_1} \cap R_{x_2}$$

MULT BY n $\quad n\cdot x[n] \Longleftrightarrow -z\,\frac{dX(z)}{dz}$

MULT BY COMPLEX EXPONENTIAL

$$z_0^n\,x[n] \Longleftrightarrow X\!\left(\frac{z}{z_0}\right), \quad R_x\,|z_0|$$

$z_0^n = \rho_0^n \cdot e^{j\omega_0 n}$

$\rho_0 = 1 \Rightarrow z_0^n = e^{j\omega_0 n}$

$\omega_0 = 0 \Rightarrow z_0^n = \rho_0^n$

ZTs + LSI SYSTEMS

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

$x[n] \Leftrightarrow X(z)$, R_x

$h[n] \Leftrightarrow H(z)$, R_h

$y[n] \Leftrightarrow Y(z)$, $R_y = R_x \cap R_h$

$$Y(z) = X(z)H(z)$$

$$H(z) = \frac{Y(z)}{X(z)}$$

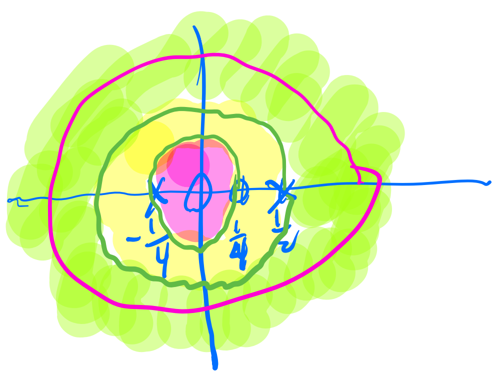
CONSIDER

$$H(z) = \frac{3 - \frac{3}{4}z^{-1}}{1 - \frac{1}{4}z^{-1} - \frac{1}{8}z^{-2}} = \frac{3\left(1 - \frac{1}{4}z^{-1}\right)}{\left(1 + \frac{1}{4}z^{-1}\right)\left(1 - \frac{1}{2}z^{-1}\right)}$$

$$= \frac{3z\left(z - \frac{1}{4}\right)}{\left(z + \frac{1}{4}\right)\left(z - \frac{1}{2}\right)}$$

ZEROS of $H(z)$: $0, 1/4$

POLES of $H(z)$: $\frac{1}{2}, -\frac{1}{4}$



POSSIBLE ROCs

1. $|z| > 1/2$ — RIGHT-SIDED, STABLE
2. $|z| < 1/4$ — LEFT-SIDED, UNSTABLE
3. $\frac{1}{4} < |z| < 1/2$ — BOTH-SIDED, UNSTABLE

ZTs + DIFFERENCE EQS...

$$H(z) = \frac{3 - \frac{3}{4}z^{-1}}{1 - \frac{1}{4}z^{-1} - \frac{1}{8}z^{-2}} = \frac{Y(z)}{X(z)}$$

$$Y(z)\left(1 - \frac{1}{4}z^{-1} - \frac{1}{8}z^{-2}\right) = X(z)\left(3 - \frac{3}{4}z^{-1}\right)$$

$$Y(z) - \frac{1}{4}z^{-1}Y(z) - \frac{1}{8}z^{-2}Y(z) = 3X(z) - \frac{3}{4}z^{-1}X(z)$$

$$y[n] - \frac{1}{4}y[n-1] - \frac{1}{8}y[n-2] = 3x[n] - \frac{3}{4}x[n-1]$$

$$y[n] = \frac{1}{4}y[n-1] + \frac{1}{8}y[n-2] + 3x[n] - \frac{3}{4}x[n-1]$$

DIFF. DIFFERENCE EQN

IN GENERAL IF $H(z) = \dfrac{\sum_{\ell=0}^{M} b_\ell z^{-\ell}}{\sum_{k=0}^{N} a_k z^{-k}} = \dfrac{Y(z)}{X(z)} = \dfrac{B(z)}{A(z)}$

THEN $$\sum_{k=0}^{N} a_k y[n-k] = \sum_{\ell=0}^{M} b_\ell x[n-\ell]$$

IF $a_0 = 1$ $$y[n] = -\sum_{k=1}^{N} a_k y[n-k] + \sum_{\ell=0}^{M} b_\ell x[n-\ell]$$

N = # POLES, "ORDER" OF SYSTEM

M = # ZERO

INVERSE Z-TRANSFORMS $\quad x[n] = \dfrac{1}{2\pi j} \oint_c X(z)\, z^{n-1}\, dz$

WAYS OF COMPUTING IZT

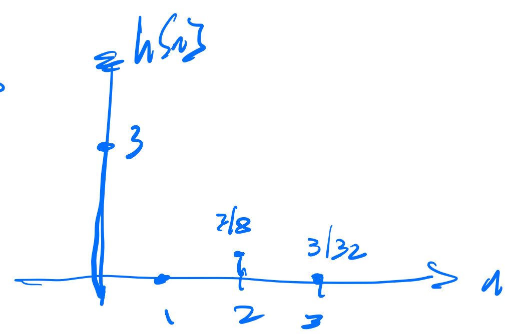
1. DIRECT COMPUTATION of CONTOUR INTEGRALS
2. LONG DIVISION !
3. ★ PARTIAL-FRACTION EXPANSION
4. TAYLOR SERIES EXPANSION
5. ITERATION
6. FORMAL SOLUTION of DIFFERENCE EQ.

$\boxed{\text{ITERATION}}$ CONSIDER $\quad y[n] = \frac{1}{4} y[n-1] + \frac{1}{8} y[n-2] + 3x[n] - \frac{3}{4} x[n-1]$

$h[n] = y[n]$ FOR $x[n] = \delta[n]$

IF $x[n] = 0$, FOR $n<0$, $y[n-1], y[n-2], y[n-3], = 0$

n	$x[n-1]$ ($-3/4$)	$x[n]$ (3)	$y[n-2]$ ($1/8$)	$y[n-1]$ ($1/4$)	$y[n]$
0	0	1	0	0	3
1	1	0	0	3	0
2	0	0	3	0	3/8
3	0	0	0	3/8	3/32



LONG DIVISION

$$H(z) = \frac{3 + \frac{3}{4}z^{-1}}{1 - \frac{1}{4}z^{-1} - \frac{1}{8}z^{-2}}$$

$$3 + 0z^{-1} + \frac{3}{8}z^{-2} + \frac{3}{32}z^{-3} + \frac{9}{128}z^{-4}$$

$$1 - \frac{1}{4}z^{-1} - \frac{1}{8}z^{-2} \;\Big)\; 3 - \frac{3}{4}z^{-1} + 0z^{-2} + 0z^{-3} + 0z^{-4}$$

$$3 - \frac{3}{4}z^{-1} - \frac{3}{8}z^{-2}$$

$$0z^{-1} + \frac{3}{8}z^{-2} + 0z^{-3}$$

$$0z^{-1} + 0z^{-2} + 0z^{-3}$$

$$\frac{3}{8}z^{-2} + 0z^{-3} + 0z^{-4}$$

$$\frac{3}{8}z^{-2} - \frac{3}{32}z^{-3} - \frac{3}{64}z^{-4}$$

$$\frac{3}{32}z^{-3} + \frac{3}{64}z^{-4} + 0z^{-5}$$

$$\frac{3}{32}z^{-3} - \frac{3}{128}z^{-4} - \frac{3}{256}z^{-5}$$

FOR LHS RESPONSES:

$$-\frac{1}{8}z^{-2} - \frac{1}{4}z^{-1} + 1 \;\Big)\; -\frac{3}{4}z^{-1} + 3 + 0z + 0z^{2} + \ldots$$