MORE DTFT PROPERTIES!

PARSEVAL'S THEOREM

$$\text{ENERGY} = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 \, d\omega$$

INITIAL VALUE THEOREMS

$$x[0] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, d\omega$$

$$\sum_{n=-\infty}^{\infty} x[n] = X(e^{j\omega})\Big|_{\omega=0} = X(e^{j0})$$

DTFT:

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} \, d\omega$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

INTRO TO Z-TRANSFORMS

(OS4P 3.0-3.1)

ZT

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

(LAPLACE TRANSFORM)

CTFT

$$\{e^{j\Omega t}\}$$

LT

$$\{e^{st}\}$$

$$s = \sigma + j\Omega$$

$$e^{(\sigma + j\Omega)t} = e^{\sigma t} \cdot e^{j\Omega t}$$

DTFT

$$\{e^{j\omega n}\}$$

ZT

$$\{z^n\}$$

$$z = \rho e^{j\omega}$$

$$z^n = \rho^n e^{j\omega n}$$

$$(-1)^n = \left(e^{j\pi}\right)^n$$

$$x[n]\, e^{j\omega_0 n} \Longleftrightarrow X\left(e^{j(\omega - \omega_0)}\right)$$

$$X(z) = \frac{1}{1-\alpha z^{-1}} = \frac{z}{z-\alpha} = \frac{B(z)}{A(z)}$$

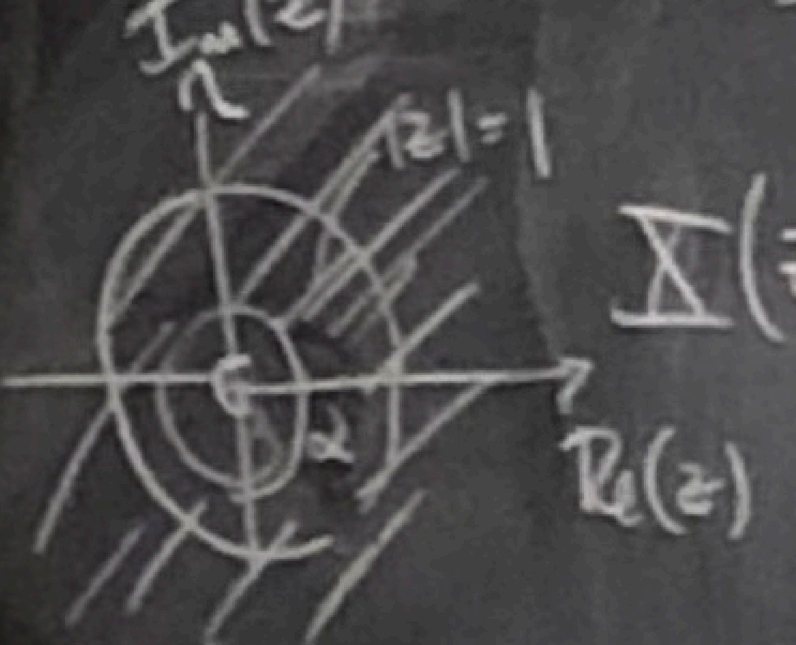
ZEROS OF $X(z)$ ARE $\{z\}$ FOR WHICH $B(z)=0$

$z=0$

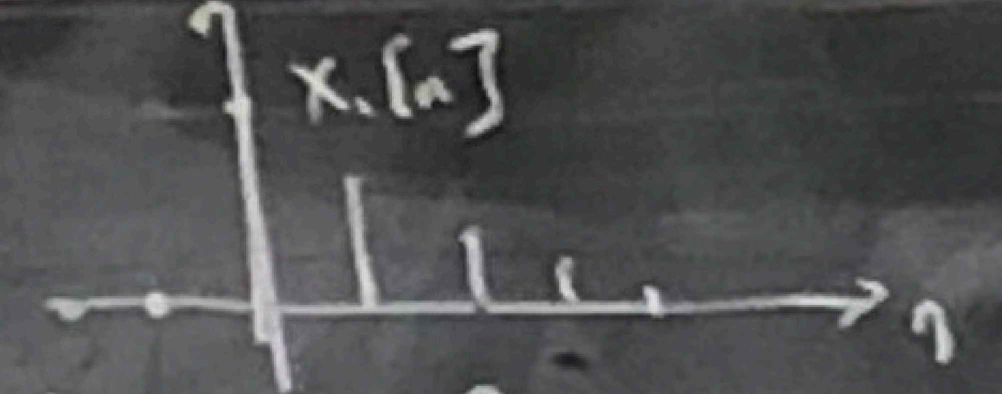
POLES OF $X(z)$ ARE $\{z\}$ FOR WHICH $A(z)=0$

$z=\alpha$

Ex 1



$x[n] = \alpha^n u[n]$



$$X(z) = \sum_{n=0}^{\infty} \alpha^n z^{-n} = \sum_{n=0}^{\infty} (\alpha z^{-1})^n = \frac{1}{1-\alpha z^{-1}}$$

FOR $|\alpha z^{-1}| < 1$, $|\alpha| < |z|$

$|z| > |\alpha|$

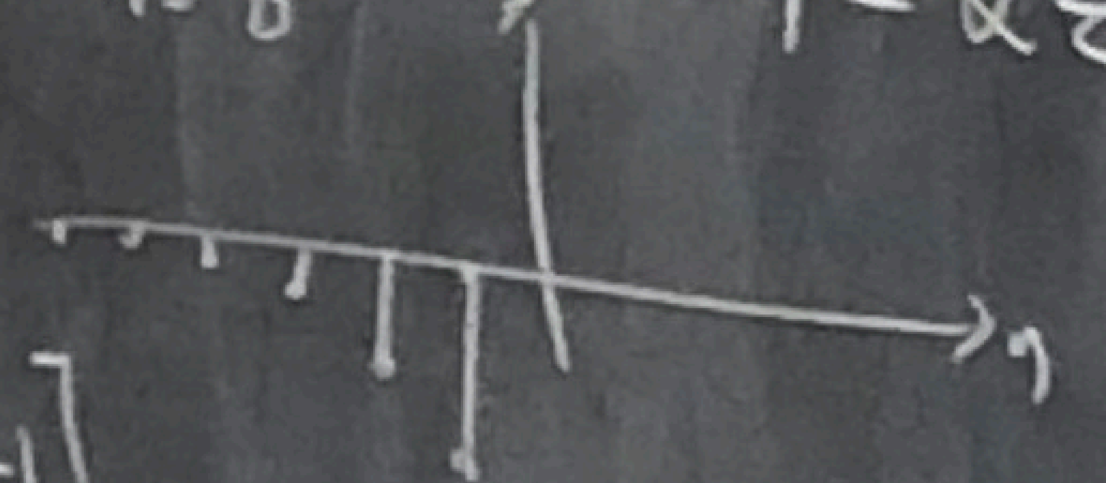
REGION OF CONVERGENCE (ROC)

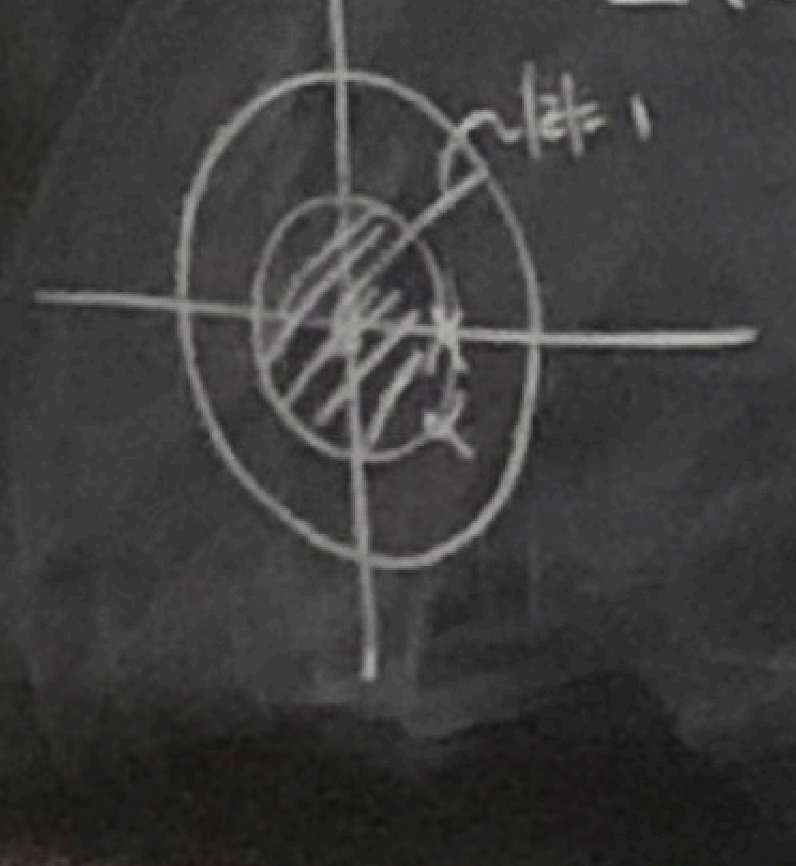
$$(-1)^n = (e^{j\pi})^n$$

$$x[n]\, e^{j\omega_0 n} \Leftrightarrow X(e^{j(\omega-\omega_0)})$$

Ex 2

$x[n] = -\alpha^n u[-n-1]$





$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = \sum_{n=-\infty}^{-1} -\alpha^n z^{-n} = \sum_{\ell=+1}^{+\infty} -\alpha^{-\ell} z^{\ell} = -\sum_{m=0}^{\infty} \alpha^{-(m+1)} z^{(m+1)} = -\alpha^{-1} z^{1} \sum_{m=0}^{\infty} (\alpha^{-1} z)^m = -\alpha^{-1} z^{1} \frac{1}{1-\alpha^{-1} z} \cdot \frac{-\alpha z^{-1}}{-\alpha z^{-1}}$$

$\ell = -n$

$n = -\ell$

$m = \ell - 1$

$\ell = m + 1$

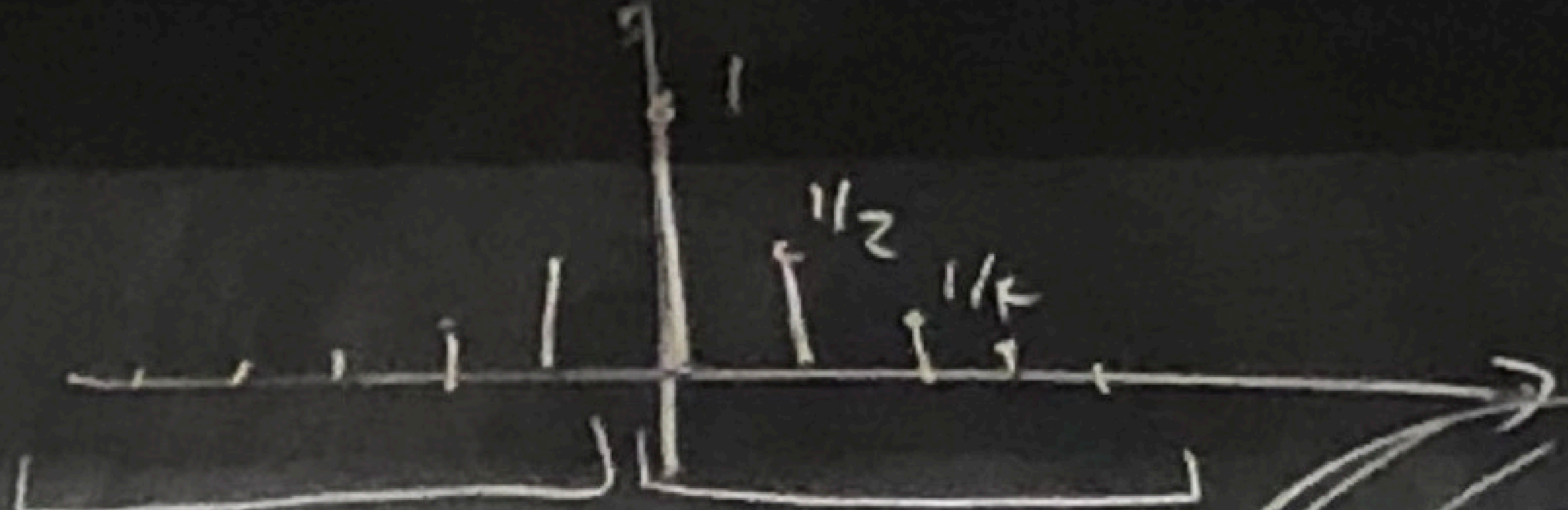
$$= \frac{1}{1-\alpha z^{-1}}$$

$|\alpha^{-1} z| < 1$; $|z| < |\alpha|$

3. BOTH-SIDED FUNCTION

DTFT

Ex $x[n] = \left(\frac{1}{2}\right)^{|n|}, \forall n$



$= \left(\frac{1}{2}\right)^n u[n] + 2^n u[-n-1]$

$X(z) = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{-1}{1-2z^{-1}}$

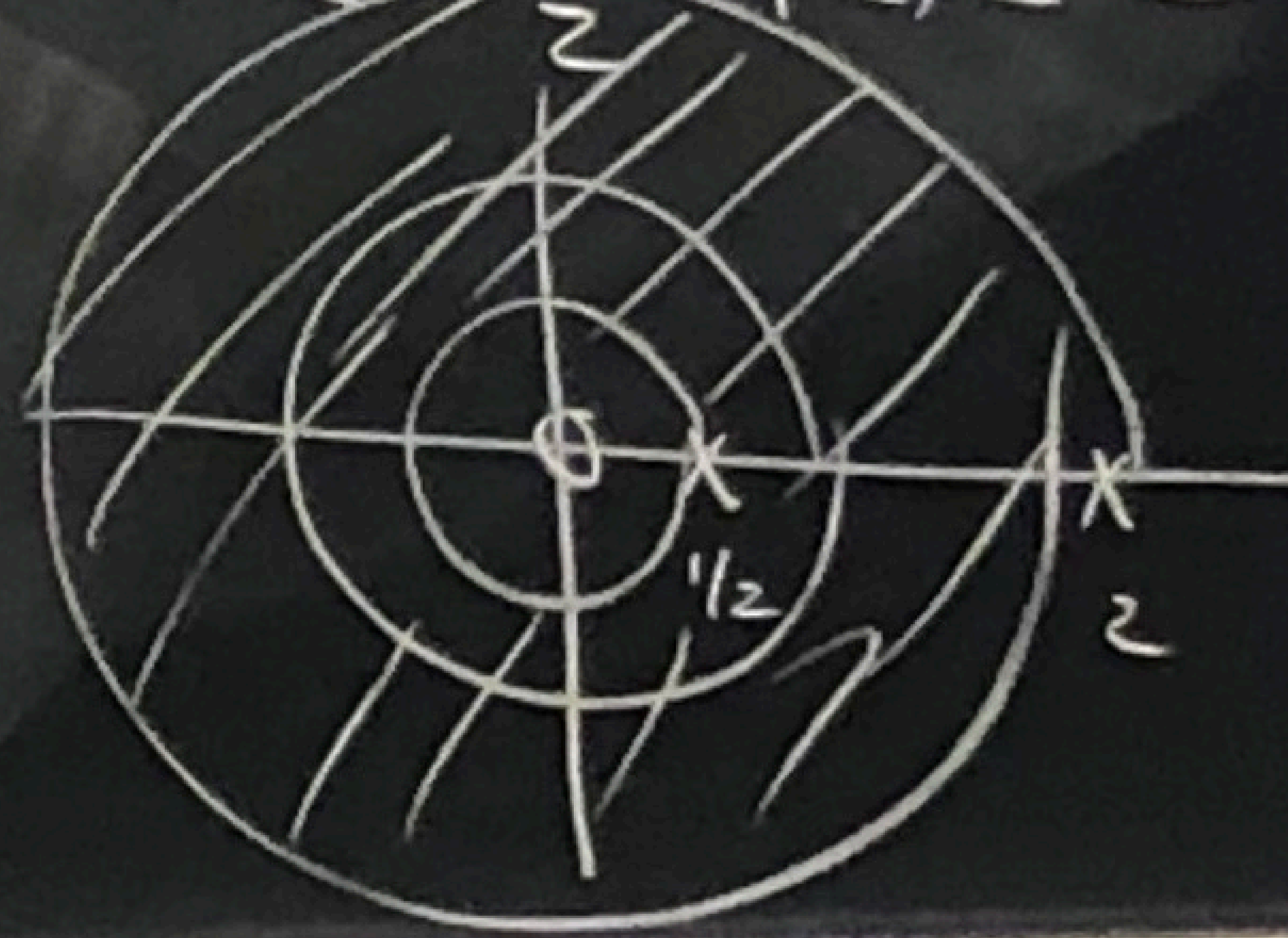
$|z| > \frac{1}{2}$ $\quad$ $|z| < 2$

ROC_1 $\quad$ ROC_2

$ROC_1 \cap ROC_2$

ANNULUS

ROC $\frac{1}{2} < |z| < 2$

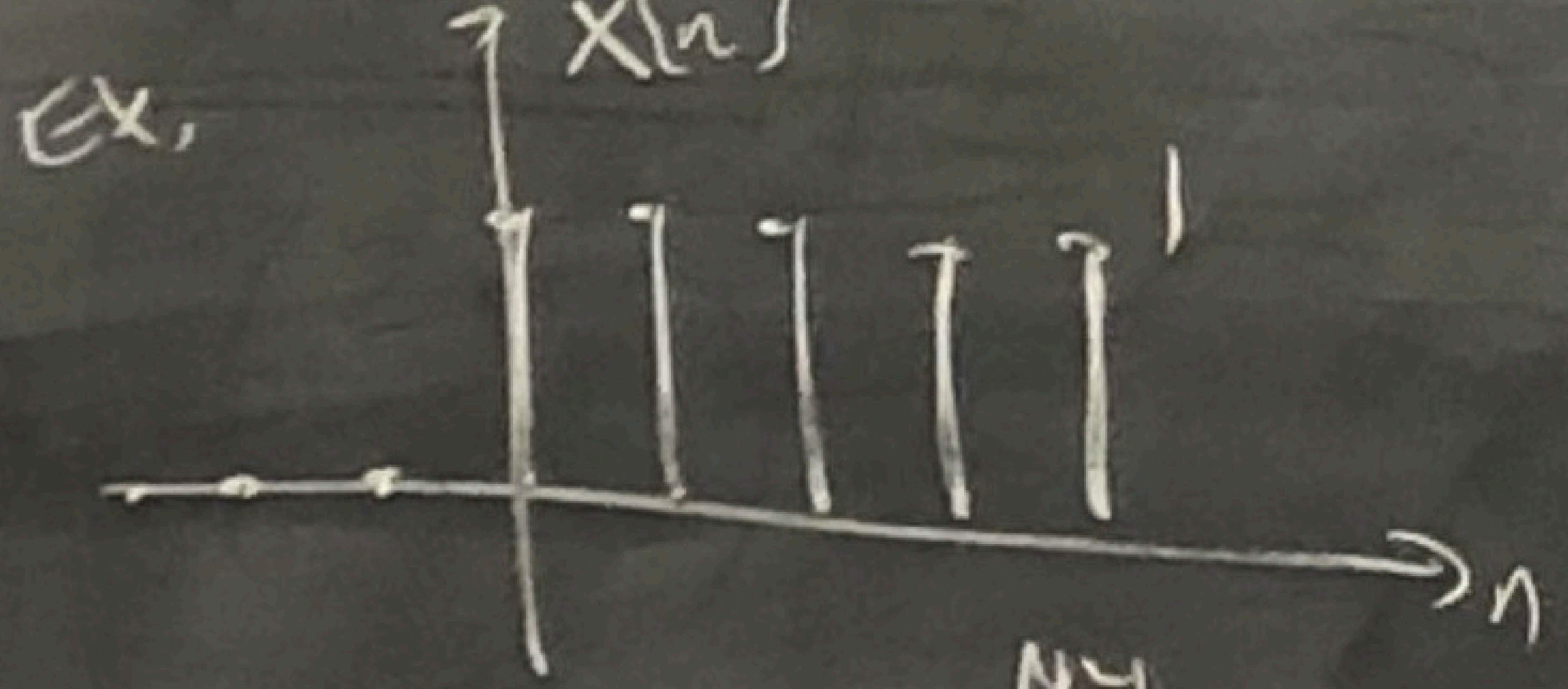


$\frac{1-2z^{-1}-1+\frac{1}{2}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-2z^{-1}\right)} = \frac{-3/2\, z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-2z^{-1}\right)} \cdot \frac{z^2}{z^2} = \frac{-3/2\, z}{\left(z-\frac{1}{2}\right)\left(z-2\right)}$

$\{e^{j\omega n}$

4. FINITE-DURATION TIME FUNCTIONS

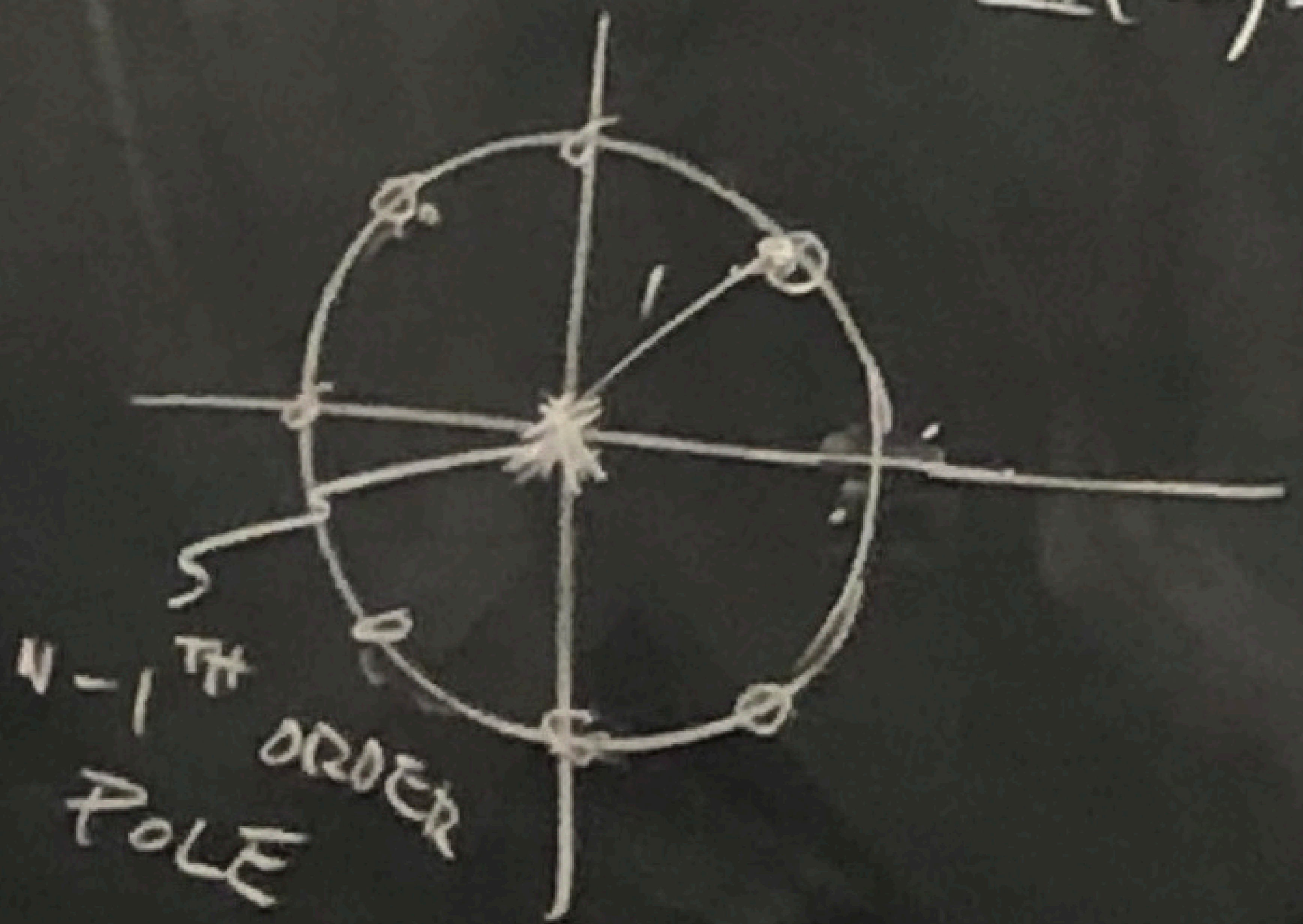
EX.



$$x[n] = \begin{cases} 1, & 0 \le n \le N-1 \\ 0, & \text{ELSE} \end{cases}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = \sum_{n=0}^{N-1} 1\cdot(z^{-1})^n = \frac{1-z^{-N}}{1-z^{-1}}$$

N = 8



$$X(z) = \frac{1-z^{-N}}{1-z^{-1}} = \frac{z^{-N}(z^N-1)}{z^{-1}(z-1)} = \frac{1}{z^{N-1}}\,\frac{(z^N-1)}{z-1}$$

$z^N = 1$

$z = 1^{1/N}$

ZT

ZTs DTFT

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$X(z)\Big|_{z=e^{j\omega}} = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$z = e^{j\omega}$

$$= X(e^{j\omega})$$

LES

IN ORDER FOR DTFT TO EXIST, ROC OF $X(z)$ MUST INCLUDE $|z|=1$

T POSSIBLY $z=0$

STABILITY OF ZT

IF $h[n]$ IS SAMPLE RESP. of LSI SYSTEM

IT CAN BE SHOWN $H(z)$ IS STABLE IFF ROC OF

INCLUDES $|z|$

IF LSI SYSTEM IS CAUSAL + STABLE

ALL POLES MUST BE INSIDE THE UNIT CIRCLE