1/29/24

INTRO TO Z-TRANSFORMS

(OSYP 3.0 - 3.2)

DTFT $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$

IF $\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$

$x[n] = \frac{1}{2\pi} \int_{\langle 2\pi \rangle} X(e^{j\omega}) e^{j\omega n} d\omega$

DTFTs $x[n]$ USING $\{e^{j\omega n}\}$

CTFTs + ~~THE~~ LAPLACE TRANSFORM

IF $\int_{-\infty}^{\infty} |x(t)|^2 dt < \infty$,

CTFT $x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\Omega) e^{j\Omega t} d\Omega$

$x(t)$ USING $\{e^{j\Omega t}\}$

$X(j\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt$

W/ THE LAPLACE TRANSFORMS

$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$

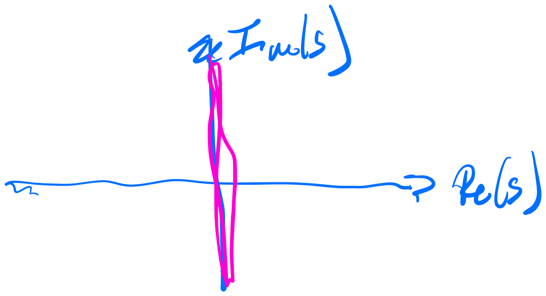
Let $\boxed{s = \sigma + j\Omega}$

$x(t) = \frac{1}{2\pi j} \oint_{C} X(s) e^{st} ds$

$x(t)$ USING $\{e^{st}\}$

$e^{st} = e^{(\sigma + j\Omega)t} = \underbrace{e^{\sigma t}}_{\mathbb{R}} \cdot \underbrace{e^{j\Omega t}}_{\mathbb{C}}$

$X(j\Omega) = X(s)\Big|_{\substack{\sigma = 0 \\ s = j\Omega}}$



THE Z-TRANSFORM

$\star$ DTFT $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$

$x[n] = \frac{1}{2\pi} \int_{\langle 2\pi \rangle} X(e^{j\omega}) e^{j\omega n} d\omega$

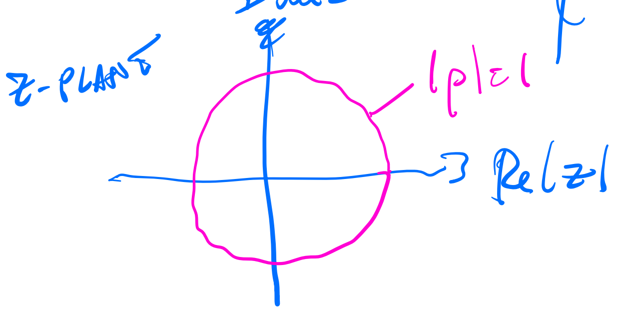
THE Z-TRANSFORM

DTFT REPRESENT $x[n]$ USING $\{e^{j\omega n}\}$

ZT REPRESENT $x[n]$ USING $\{z^n\}$

Let $\boxed{z = \rho e^{j\omega}}$

$z^n = (\rho e^{j\omega})^n = \rho^n e^{j\omega n}$



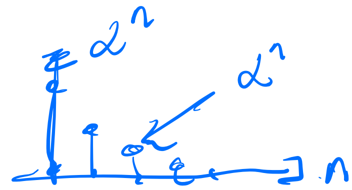
ZT $X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$

$x[n] = \frac{1}{2\pi j} \oint_c X(z) z^{n-1} dz$

$X(e^{j\omega}) = X(z)\Big|_{z=e^{j\omega}}$

EXAMPLES of ZTs...

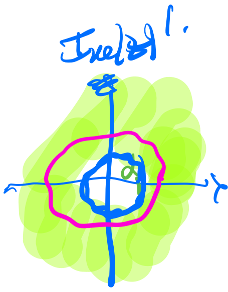
1. $x_1[n] = \alpha^n u[n]$



FOR $|\alpha z^{-1}| < 1$

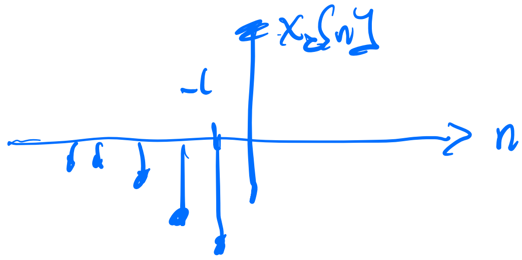
$|z| > |\alpha|$

REGION OF CONVERGENCE (ROC)



$X_1(z) = \sum_{n=-\infty}^{\infty} x_1[n] z^{-n} = \sum_{n=0}^{\infty} \alpha^n z^{-n} = \sum_{n=0}^{\infty} (\alpha z^{-1})^n = \frac{1}{1-\alpha z^{-1}}$

2. $x_2[n] = -\alpha^n u[-n-1]$



$X_2(z) = \sum_{n=-\infty}^{\infty} x_2[n] z^{-n}$

$= \sum_{n=-\infty}^{-1} -\alpha^n z^{-n} = \sum_{\ell=1}^{\infty} -(\alpha)^{-\ell} z^{\ell} = \sum_{n=0}^{\infty} -\alpha^{-(n+1)} z^{n+1}$

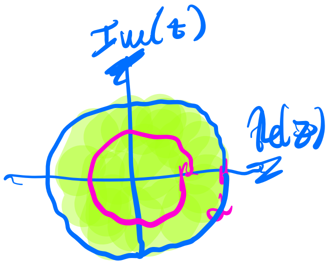
$\ell = -n$

$n = -\ell$

$\sum_{n=\ell-1}$

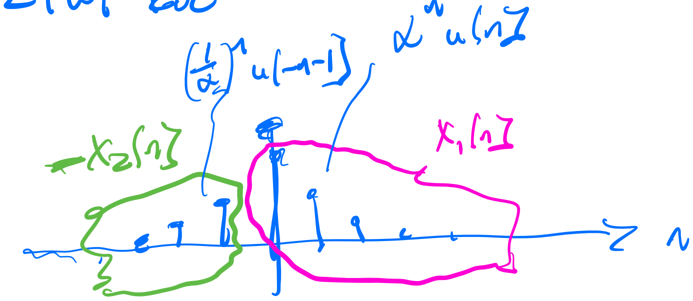
$\ell = n+1$

$$X_2(z) = -\alpha^{-1}z \sum_{n=0}^{\infty} \left(\alpha^{-1}z\right)^n = -\alpha^{-1}z \frac{1}{1-\alpha^{-1}z} = \frac{1}{-\alpha z^{-1}+1} = \frac{1}{1-\alpha z^{-1}}$$



$\left|\alpha^{-1}z\right| < 1$, $|z| < |\alpha|$ ROC

EX 3. $x_3[n] = \alpha^{|n|}$, $|\alpha| < 1$

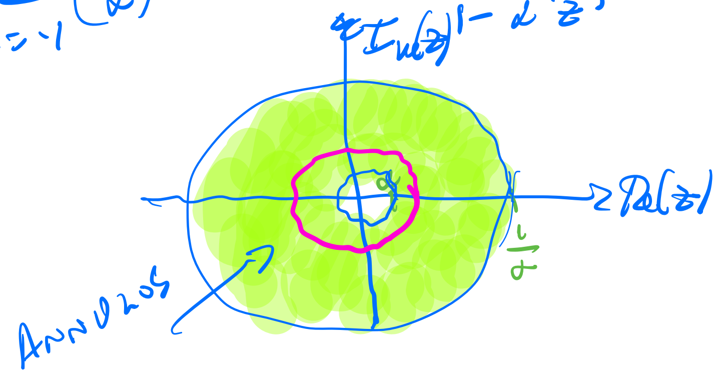


$$X_3(z) = \underbrace{X_1(z)}_{|z|>|\alpha|} - \underbrace{X_2(z)}_{|z|<\left|\frac{1}{\alpha}\right|}$$

$$= \frac{1}{1-\alpha z^{-1}} - \frac{1}{1-\frac{1}{\alpha}z^{-1}} = \frac{1-\alpha^{-1}z^{-1}-1+\alpha z^{-1}}{(1-\alpha z^{-1})(1-\alpha^{-1}z^{-1})}$$

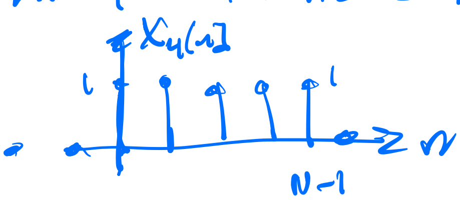
$$\sum_{n=-\infty}^{-1} \left(\frac{1}{\alpha}\right)^n z^{-n} = -\frac{1}{1-\alpha^{-1}z^{-1}}$$

$$= \frac{\left(\alpha - \frac{1}{\alpha}\right)z^{-1}}{(1-\alpha z^{-1})(1-\alpha z^{-1})}$$



ROC $|\alpha| < |z| < \left|\frac{1}{\alpha}\right|$

EX 4 FINITE-DURATION



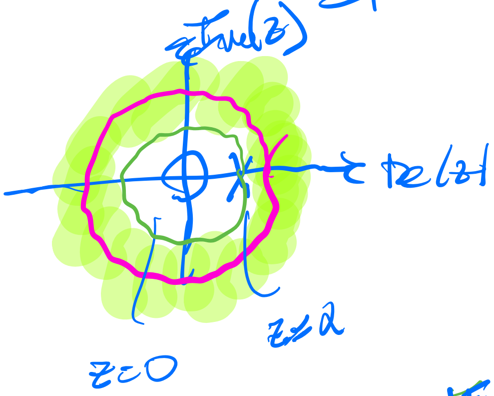
$$x_4[n] = \begin{cases} 1, & 0 \le n \le N-1 \\ 0, & \text{ELSE} \end{cases}$$

$$X(z) = \sum_{n=-\infty}^{\infty} \left(u[n] - u[n-N]\right) z^{-n} = \sum_{n=0}^{N-1} 1 \cdot z^{-n} = \frac{1-z^{N}}{1-z^{-1}}$$

ROC $= \forall z$

Poles + Zeros

Ex. 1 $X_1(z) = \dfrac{1}{1-\alpha z^{-1}} = \dfrac{z}{z-\alpha}$



Zero @ $z=0$

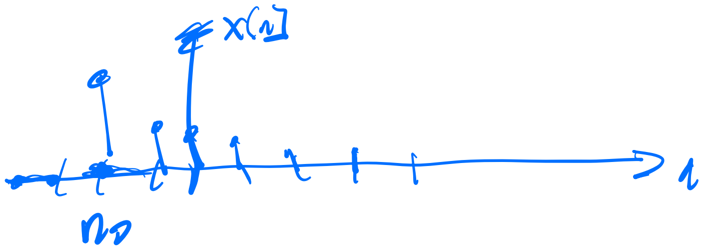
Pole @ $z=\alpha$

If $X(z) = \dfrac{N(z)}{D(z)}$ Zeros are roots of $N(z)$ $z=0$

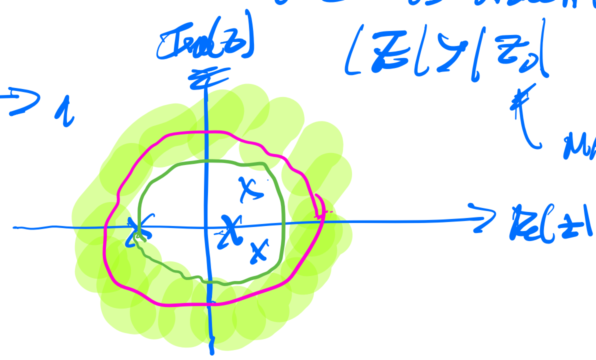
Poles are roots of $D(z)$ $z=\alpha$

Types of functions + their ROCs

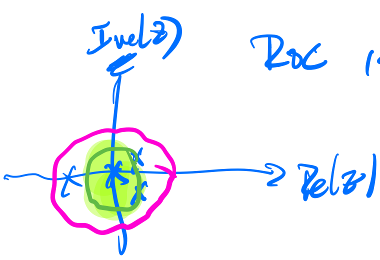
1. Right-sided function $x[n]=0,\ n<n_0$



ROC is always of form $|z|>|z_0|$ ← magnitude of farthest pole from origin



2. Left-sided function $x[n]=0,\ n\geq n_0$



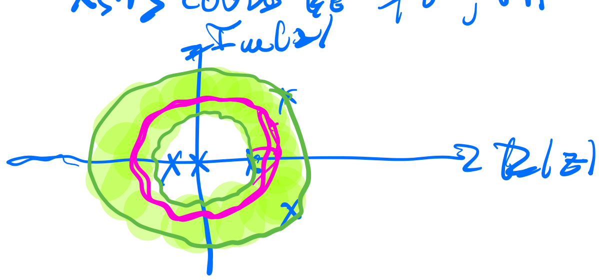
ROC is of form $|z|<|z_0|$, except maybe not $z=0$

magnitude of closest pole to origin excluding $z=0$

3. "Both-sided" function

ROC $|z_1|<|z|<|z_2|$

$x[n]$ could be $\neq 0$, $\forall n$



4. FINITE-DURATION FN. $x[n]$

ROC IS ENTIRE Z-PLANE POSSIBLY EXCLUDING $z=0$

PROPERTIES OF Z-TRANSFORMS

1. LINEARITY

$$a x_1[n] + b x_2[n] \Leftrightarrow a X_1(z) + b X_2(z), \quad \text{ROC } R_{x_1} \cap R_{x_2}$$

2. TIME SHIFT

$$x[n-N] \Leftrightarrow \sum_{n=-\infty}^{\infty} x[n-N] z^{-n} = \sum_{\ell=-\infty}^{\infty} x[\ell] z^{-(\ell+N)}$$

Let $\ell = n-N$

$n = \ell + N$

$$= z^{-N} \cdot \underbrace{\sum_{\ell=-\infty}^{\infty} x[\ell] z^{-\ell}}_{X(z)}$$

$$x[n-N] \Leftrightarrow z^{-N} \cdot X(z)$$

ROC $= R_x$